Problem On Motion Of Rigid Bodies In Non-Newtonian Incompressible Fluid *

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The first part of the present work is devoted to justification of an existence theorem for a problem on motion of non-Newtonian incompressible fluid possessing nonhomogeneous viscosity properties (the viscosity function is supposed to be transported along with particles of fluid). In the second part of the work, we prove that, if the initial values of viscosity tend to infinity in some subdomains, then generalized solution of the original problem tends to generalized solution of a problem on motion of rigid bodies in the fluid, where motion of bodies is controlled by hydrodynamic reactions.

1 Formulations Of Problems

The problem on a motion of non-Newtonian viscous incompressible fluid is as follows.

Problem A. Fluid occupies a bounded domain Ω in $\mathbb{R}^3$. We seek for velocity field $\vec{u} : Q_T \to \mathbb{R}^3$, pressure $p_* : Q_T \to \mathbb{R}$ and viscosity $\mu : Q_T \to \mathbb{R}$, $Q_T = \Omega \times [0,T]$ satisfying the following equations and initial and boundary conditions:

$$D_t\vec{u} + \textstyle\sum_{i=1}^3 u_i D_i \vec{u} - \mathrm{div}(\mu W) = \vec{f} - \nabla p_*, \quad (x,t) \in Q_T, \tag{1.1}$$

$$D_t\mu + \textstyle\sum_{i=1}^3 u_i D_i \mu = 0, \quad (x,t) \in Q_T, \tag{1.2}$$

$$\mathrm{div}\vec{u} = 0, \quad (x,t) \in Q_T, \tag{1.3}$$

$$\vec{u}(x,t)|_{t=0} = \vec{u}_0(x), \quad x \in \Omega, \tag{1.4}$$

$$\vec{u}(x,t)|_{\partial\Omega} = 0, \quad \mu(x,t)|_{t=0} = \mu_0(x), \tag{1.5}$$

$$0 < \tilde{m} \le \mu_0(x) \le \tilde{M} < \infty, \quad \tilde{m}, \tilde{M} = \mathrm{const}\ , \ x \in \Omega. \tag{1.6}$$

W is supposed to satisfy the following demands: $\mu W \in \partial\Phi(\mathbb{D}(\vec{u}))$, $\partial\Phi(\mathbb{D}(\vec{u}))$ is the subdifferential of the functional $\Phi(\chi) = \frac{1}{p}\int_\Omega Q(x,\chi(x))dx$ at a point $\chi = \mathbb{D}(\vec{u})$, where

$$Q(x,\chi(x)) = \begin{cases} \mu|\chi|^p, & \text{if } |\chi| \le M \\ +\infty, & \text{if } |\chi| > M,\ M = \mathrm{const} < +\infty. \end{cases}$$

In formulae (1.1)–(1.6) and in the rest of the paper the following notations are in use: $D_i = \partial/\partial x_i$, $D_t\vec{\varphi} = \partial\vec{\varphi}/\partial t$, $(\vec{u} \otimes \vec{v})_{ij} = u_i v_j$, $(\nabla\vec{u})_{ij} = D_i u_j$, $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$ is a tensor product of two matrices; $D_{ij}(\vec{u}) = (1/2)(D_i u_j + D_j u_i)$ is deformation tensor, $\vec{a}(\mu,\vec{u}) = \{-\sum_{i=1}^3 D_i(\mu|\mathbb{D}(\vec{u})|^{p-2} D_{ij}(\vec{u}))\}_{j=1}^3$, $|\mathbb{D}(\vec{u})|^2 = \mathbb{D}(\vec{u}) : \mathbb{D}(\vec{u})$.

Also we use the following functional spaces: $\mathcal{D}(\Omega)$ is the space of infinitely smooth compactly supported in Ω functions, $\mathcal{V}(\Omega) = \{\vec{\varphi} |\ \varphi_i \in \mathcal{D}(\Omega),\ \mathrm{div}\vec{\varphi} = 0\}$, $V(\Omega)$, $V^k(\Omega)$ are the closures of $\mathcal{V}(\Omega)$ with respect to norms $(W_p^1(\Omega))^3$, $(H^k(\Omega))^3$, $k = 1, 2, \ldots$, respectively, $H(\Omega)$ is the closure of $\mathcal{V}(\Omega)$ with respect to norm $(L_2(\Omega))^3$; $H_0(\Omega) = \{\vec{v} \in H(\Omega) |\ \ \vec{v}|_{\partial\Omega} = 0\}$ (here $\vec{v}|_{\partial\Omega} = 0$ is the trace of $\vec{v}$ on a boundary of domain Ω); $V'(\Omega)$, $V^{-k}(\Omega)$ are the spaces conjugate to $V(\Omega)$ and $V^k(\Omega)$ (we denote $(H_0(\Omega))^* = H_0(\Omega)$).

Definition 1.1. By a generalized solution of Problem A we call a pair of functions $\{\vec{u}(x,t),\ \mu(x,t)\}$ such that $\vec{u} \in L_p(0,T;V) \cap L_\infty(0,T;H_0(\Omega))$, $|\mathbb{D}(\vec{u}(t))| \le M$ for a. e. $t \in [0,T]$, $\mu \in L_\infty(Q_T)$, $\tilde{m} \le \mu(x,t) \le \tilde{M}$ for

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a. e. $(x,t)\in Q_T$, and the variational inequality

$$\int_{Q_T} D_t\vec\varphi(\vec\varphi-\vec u)dxdt+\int_{Q_T}\mu|\mathbb{D}(\vec u)|^{p-2}\mathbb{D}(\vec u):\mathbb{D}(\vec\varphi-\vec u)dxdt$$
$$-\int_{Q_T}\vec u\otimes\vec u:\nabla(\vec\varphi-\vec u)dxdt\ge\int_{Q_T}\vec f(\vec\varphi-\vec u)dxdt \quad (1.7)$$

and integral identity

$$\int_{Q_T}\mu(D_t\psi+\vec u\nabla\psi)dxdt+\int_\Omega\mu(x,0)\psi(x,0)dx=0, \quad (1.8)$$

hold for all test functions $\vec\varphi\in L^1_p(0,T;V)$ and $\psi\in C^1(Q_T)$ satisfying $|\mathbb{D}(\vec\varphi)|\le M$, $\vec\varphi|_{t=0}=\vec u_0$, $\psi|_{t=T}=0$.

In the paper, the global existence of a generalized solution of Problem A is proved. After that, we provide Problem A with the initial data for viscosity μ in a special form

$$\mu_\varepsilon(x,0)=\begin{cases}1, & x\in\Omega\setminus V_0,\\ 1/\varepsilon, & x\in V_0.\end{cases}$$

Here V_0 is union of non-intersecting (by pairs) subdomains $V_0^{(l)},\ l=1,\dots,N$ with smooth (Lipschitz) boundaries $\Sigma_0^{(l)}$.

Then we prove that the sequence of generalized solutions of Problem A tends as $\varepsilon\to0$ to a generalized solution of the following problem on motion of solid bodies in non-Newtonian fluid governed by hydrodynamic reactions.

Problem B. It is necessary to find locations of domains $V^{(l)}(t),\ l=1,\dots,N$, occupied by solid bodies, velocity field $\vec u:\ Q_T\backslash V_T\to R^3$ and pressure $p_*:\ Q_T\backslash V_T\to R$ in the liquid component (here $V_T=\{(V(t),t),\ t\in[0,T]\},\ V(t)=\bigcup\limits_{l=1}^{N}V^{(l)}(t)$), satisfying the system which is composed of the equations describing motion of fluid

$$D_t\vec u+\textstyle\sum_{i=1}^3u_iD_i\vec u-\mathrm{div}W=\vec f-\nabla p_*,\quad (x,t)\in Q_T\setminus V_T, \quad (1.9)$$
$$\mathrm{div}\vec u=0,\quad (x,t)\in Q_T\setminus V_T, \quad (1.10)$$

and of Euler equations that describe a motion of solid bodies governed by hydrodynamics reactions and are written in inertial (immovable) Descartes coordinate system [1]:

$$m^{(l)}\frac{d\vec v_c^{(l)}}{dt}=\int_{V^{(l)}}\rho^{(l)}\vec f dx+\int_{\Sigma^{(l)}}\mathbb{T}\vec n d\sigma, \quad (1.11)$$

$$m^{(l)}x_c^{(l)}\times\frac{d\vec v_c^{(l)}}{dt}+(J^{(l)}+2m^{(l)}x_c^{(l)}\vec v_c^{(l)}I-m^{(l)}\vec v_c^{(l)}\otimes x_c^{(l)}$$
$$-m^{(l)}x_c^{(l)}\otimes\vec v_c^{(l)})\vec\omega^{(l)}+(J^{(l)}+m^{(l)}x_c^{(l)2}-m^{(l)}x_c^{(l)}\otimes x_c^{(l)})\frac{d\vec\omega^{(l)}}{dt}$$
$$=\int_{V^{(l)}}\rho^{(l)}(x\times\vec f)dx+\int_{\Sigma^{(l)}}x\times(\mathbb{T}\vec n)d\sigma,\quad l=1,\dots,N. \quad (1.12)$$

Remark 1.1. Viscosity of fluid is constant equal to 1 in the whole domain occupied by fluid.

Equations (1.9)–(1.12) are supplemented by initial and boundary conditions

$$V^{(l)}(0)=V_0^{(l)},\quad l=1,\dots,N, \quad (1.13)$$
$$\vec u(x,t)|_{t=0}=\vec u_0(x),\quad x\in\Omega\setminus V_0, \quad (1.14)$$
$$\vec u(x,t)|_{\partial\Omega}=0, \quad (1.15)$$
$$\vec u(x,t)|_{\partial V_T}=(\vec v_c(t)+\vec\omega(t)\times(x-x_c(t)))|_{\partial V_T}, \quad (1.16)$$
$$\vec v_c|_{t=0}=\vec a_c,\quad \vec\omega|_{t=0}=\vec\omega_0,\quad x_c|_{t=0}=x_c^0,\quad x\in V_0. \quad (1.17)$$

In (1.11)–(1.17) $\Sigma^{(l)}$ is a surface of a solid body $V^{(l)}$, $\vec n$ is the outward with respect to $V^{(l)}$ normal vector to $\Sigma^{(l)}$, $\rho^{(l)}\ge0$ is density distribution in a solid body, in this paper we set $\rho^{(l)}=1$ for all $l,\ 1\le l\le N$; $m^{(l)}>0$ is mass of a solid body, $J^{(l)}$ is inertia tensor of body related to main axes of inertia, $J_{ij}^{(l)}=\delta_{ij}\int_{V^{(l)}}\rho^{(l)}(x_{i-1}^2+x_{i+1}^2)dx,\ i,j=1,2,3;$ $\vec v_c^{(l)}$ is velocity of inertia centre of a body, $x_c^{(l)}$ is vector-radius of inertia centre of a body,

$\vec{\omega}^{(l)}$ is angular velocity of a body, $\mathbb{T}$ is the stress tensor in viscous fluid, $T_{ij} = -p_*\delta_{ij} + W_{ij}$, p_* is the inner pressure in fluid, $\vec{f}$ is vector of external mass forces.

In order to formulate the generalized solution of Problem B, let us introduce the function characterizing locations of solid bodies:

$$\Lambda(x,t) = \begin{cases} 1, & x \in V(t), \\ 0, & x \in \Omega \setminus V(t). \end{cases}$$

In terms of this function the initial condition (1.13) may be rewritten in form

$$\Lambda(x,0) = \Lambda_0(x) = \begin{cases} 1, & x \in V_0, \\ 0, & x \in \Omega \setminus V_0. \end{cases} \tag{1.18}$$

In line with [2], also introduce several special classes of functions:

$$\mathrm{Char}(E) \text{ is the class of characteristic functions of subsets of a set } E,$$
$$K_\delta(\chi) = \{\vec{\psi} \in H_0^1(\Omega) \,|\, \mathbb{D}(\vec{\psi})(x) = 0,\ x \in S_\delta(\chi)\}, \quad K_0(\chi) = \overline{\cup_{\delta>0} K_\delta(\chi)},$$
$$K(\chi) = \{\vec{\psi} \in H_0^1(\Omega) \,|\, \mathbb{D}(\vec{\psi})(x) = 0,\ x \in S(\chi)\},$$

where $\chi \in \mathrm{Char}(\Omega)$, $S(\chi) = \{x \in \Omega \,|\, \chi(x) = 1\}$, S_δ is δ-neighbourhood of a set S.

Definition 1.2. By a generalized solution of Problem B we call a pair of functions $\{\vec{u}, \Lambda\}$ such that $\vec{u} \in L_\infty(0,T;H) \cap L_p(0,T;V)$, $\vec{u} \in K(\Lambda)$, $|\mathbb{D}(\vec{u}(t))| \le M$ for a. e. $t \in [0,T]$; $\Lambda \in \mathrm{Char}(Q_T)$; $\Lambda \in C(0,T;L_\vartheta(\Omega))$, $\vartheta < \infty$, and for which the integral inequality

$$\int_{Q_T} D_t\vec{\varphi}(\vec{\varphi} - \vec{u})dxdt + \int_{Q_T} |\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}) : \mathbb{D}(\vec{\varphi} - \vec{u})dxdt$$
$$- \int_{Q_T} \vec{u} \otimes \vec{u} : \nabla(\vec{\varphi} - \vec{u})dxdt \ge \int_{Q_T} \vec{f}(\vec{\varphi} - \vec{u})dxdt \tag{1.19}$$

and the integral identity

$$\int_{Q_T} \Lambda(D_t\psi + \vec{u}\nabla\psi)dxdt + \int_\Omega \Lambda_0\psi(x,0)dx = 0 \tag{1.20}$$

hold. Here $\vec{\varphi} \in W_p^1(0,T;V)$ is a test vector field satisfying the conditions $\vec{\varphi} \in K(\Lambda)$, $|\mathbb{D}(\vec{\varphi})| \le M$, $\vec{\varphi}|_{t=0} = \vec{u}_0$, and $\psi \in C^1(Q_T)$ is a test function such that $\psi|_{t=T} = 0$.

Remark 1.2. The demand $\vec{u} \in K(\Lambda)$ imposed in Definition 1.2 provides that evolution in time of $V^{(l)}$, $l = 1, \ldots, N$ appears to be a motion of a solid body because any solution of equation $\mathbb{D}(\vec{u})(x) = 0$ has a form $\vec{u}(x) = \vec{v}_c + \vec{\omega} \times \vec{x}$ where $\vec{v}_c$ and $\vec{\omega}$ do not depend on x [3, Chapter III, §2.1].

Remark 1.3. The class of test functions for Inequality (1.19) depends on a solution of the problem. From the justification of existence theorem for Problem B one will see that such choice of the class of test functions is consistent and clear. Also, it is worth to mention that constructions of test functions depending on solutions have already been considered in [2,4].

Remark 1.4. Some explanations concerning Definitions 1.1 and 1.2 and observations concerning interactions between solid bodies and a solid body and $\partial\Omega$ will be given in Appendix.

2 Existence Of Generalized Solution To Problem A

Theorem 2.1. . *Let $\vec{f} \in L^{p'}(0,T;V')$, $|\mathbb{D}(\vec{F})| \le M$ where $D_t\vec{F} = \vec{f}$,*

$$\vec{u}_0 \in H, \quad |\mathbb{D}(\vec{u}_0)(x)| \le M, \quad p^{-1} + (p')^{-1} = 1, \quad p \ge \frac{11}{5}. \tag{2.1}$$

Then there exists a generalized solution to Problem A.

Proof. Verification of Theorem 2.1 consists of two stages. First, we formulate an auxiliary problem that involves penalty term and then prove its solvability. Second, we justify that the sequence of solutions of the auxiliary problem tends to a solution of Problem A.

2.1 Auxiliary problem A_ε

Consider the set $L = \{\varphi | \varphi \in V, |\mathbb{D}(\varphi)| \le M, M = const\}$. Note that L is the closed convex bounded set in V containing zero point. Consider the penalty operator $\beta(\vec{v}) = \{\beta_1(\vec{v}), \beta_2(\vec{v}), \beta_3(\vec{v})\}$, where $\beta_j(\vec{v}) = -\sum_{i=1}^n D_i\{([|\mathbb{D}(\vec{v})|^2 - M^2]^+)^{\frac{p}{2}-1} D_{ij}(\vec{v})\}$. Here,

$$a^+ = \begin{cases} a, & \text{if } a > 0, \\ 0, & \text{if } a \le 0. \end{cases}$$

$\beta(\vec{v})$ is monotonous operator since it is the gradient of a convex functional

$$\vec{v} \to \int_\Omega ([|\mathbb{D}(\vec{v})|^2 - M^2]^+)^{p/2} dx.$$

Also note that $\beta(\vec{v})$ *is associated* with the set L, i. e. $\{\vec{v} \,|\, \vec{v} \in V,\ \beta(\vec{v}) = 0\} = L$.

Lemma 2.1. *Let functions $\vec{f}$, $\vec{u}_0$, μ_0 and exponent p be satisfying the conditions in formulations of Theorem 2.1 and Problem A. Then for any fixed $\varepsilon > 0$ there exists pair of functions $\{\vec{u}(x,t), \mu(x,t)\}$ such that*

$$\vec{u} \in L^p(0,T;V) \cap L^\infty(0,T;H_0(\Omega)), \quad \vec{u}' \in L_{p'}(0,T;V'), \quad \mu \in L_\infty(Q_T), \quad \tilde{m} \le \mu(x,t) \le \tilde{M} \text{ a. e. in } Q_T,$$

and the integral identities

$$\int_\Omega D_t\vec{u}\,\vec{v}dx + \int_\Omega \vec{a}(\mu,\vec{u})\vec{v}dx + \int_\Omega \vec{u}\otimes\vec{v} : \nabla\vec{u}dx + \tfrac{1}{\varepsilon}\int_\Omega \beta(\vec{u})\vec{v}dx = \int_\Omega \vec{f}\vec{v}dx \text{ for a. e. } t \in [0,T], \tag{2.2}$$

$$\int_{Q_T} \mu(D_t\psi + \vec{u}\nabla\psi)dxdt + \int_\Omega \mu_0\psi(x,0)dx = 0 \tag{2.3}$$

hold. Here, $\vec{v}$, ψ are test functions satisfying $\vec{v} \in V$, $\psi \in C^1(Q_T)$, $\psi|_{t=T} = 0$.

Remark 2.1. Condition $|\mathbb{D}(\vec{u}_0)(x)| \le M$ is not used in the forthcoming proof of Lemma 2.1. Consequently, it may be dropped in the formulation of the lemma.

Proof of Lemma 2.1 is based on utilization of Galerkin method. Let $\{\vec{w}_j\}$ be the total orthonormal basis in $V^3(\Omega) \cap H_0(\Omega)$. Let a solution of the following equations (composing the Galerkin system) be standing for an approximate solution $\vec{u}_m(t)$, $\mu_m(t)$ of (2.2)–(2.3).

$$\int_\Omega D_t\vec{u}_m\,\vec{w}_jdx + \int_\Omega \vec{u}_m(t)\otimes\vec{w}_j : \nabla\vec{u}_m(t)dx + \int_\Omega \vec{a}(\mu_m,\vec{u}_m(t))\vec{w}_jdx + \tfrac{1}{\varepsilon}\int_\Omega \beta(\vec{u}_m(t))\vec{w}_jdx = \int_\Omega \vec{f}(t)\vec{w}_jdx \tag{2.4}$$

$$\text{where } \vec{u}_m(t) = \sum_{k=1}^m c_{mk}(t)\vec{w}_k(x),\ 1 \le j \le m. \tag{2.5}$$

Here, $c_{mk}(t) \in C^1([0,T])$ are unknown coefficients that should be defined.

$$D_t\mu_m + \sum_{i=1}^3 u_{mi}D_i\mu_m = 0, \tag{2.6}$$

$$\mu_m|_{t=0} = \mu_{0m}(x), \tag{2.7}$$

where $\mu_{0m}(x) \in C^1(\Omega)$, $\mu_{0m} \to \mu_0$ in $L_\vartheta(\Omega)$, $\vartheta < \infty$ is arbitrary, $\tilde{m} \le \mu_{0m} \le \tilde{M}$,

$$\vec{u}_m(0) = \vec{u}_{0m} = \sum_{j=1}^m c_j\vec{w}_j(x), \quad \vec{u}_{0m} \to \vec{u}_0 \text{ in } H. \tag{2.8}$$

In order to justify solvability of Galerkin system and to pass to a limit as $m \to \infty$ we need to obtain some a priori estimates for solutions of (2.4)–(2.8).

At first, since $\vec{w}_j \in V^3(\Omega)$ due to Sobolev embedding theorem one has $\vec{w}_j \in C_0^1(\Omega)$. Hence $\vec{u}_m \in C^1(Q_T)$ and, consequently, solution $\mu_m(x,t)$ of (2.6) has the representation [5] $\mu_m(x,t) = \mu_{0m}(\xi_m(\tau,x,t)|_{\tau=0})$, where $\xi_m(\tau,x,t)$ is the solution of the Cauchy problem $\frac{d\xi_m}{d\tau} = \vec{u}_m(\xi_m,\tau)$, $\xi|_{\tau=t} = x$. Since $\tilde{m} \le \mu_{0m} \le \tilde{M}$ due to this representation the following bound is valid

$$0 < \tilde{m} \le \mu_m(x,t) \le \tilde{M} < \infty \quad (x,t) \in Q_T. \tag{2.9}$$

Next,

$$\int_\Omega \vec{a}(\mu_m,\vec{u}_m)\vec{u}_mdx = \frac{1}{2}\sum_{i,j=1}^n \int_\Omega \mu_m|\mathbb{D}(\vec{u}_m)|^{p-2}(D_i\vec{u}_{mj} + D_j\vec{u}_{mi})D_i\vec{u}_{mj}dx$$

$$= \int_\Omega \mu_m|\mathbb{D}(\vec{u}_m)|^pdx \ge \tilde{m}\int_\Omega |\mathbb{D}(\vec{u}_m)|^pdx,$$

$$\int_\Omega \beta(\vec{u}_m)\vec{u}_m dx = \int_\Omega ([|\mathbb{D}(\vec{u}_m)|^2 - M^2]^+)^{\frac{p}{2}-1}|\mathbb{D}(\vec{u}_m)|^2 dx \geq 0, \quad \int_\Omega \vec{u}_m \otimes \vec{u}_m : \nabla\vec{u}_m = 0.$$

Multiplying j-th equation in (2.4) over $c_j(t)$, summing over all $j = 1, \ldots, m$, and taking three latter expressions into account we deduce

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}_m(t)\|^2_{2,\Omega} + \tilde{m}\int_\Omega |\mathbb{D}(\vec{u}_m(t))|^p dx \leq \int_\Omega \vec{f}(t)\vec{u}_m(t)dx. \tag{2.10}$$

For any $\vec{v} \in W^1_p(\Omega)$ such that $\vec{v}|_{\partial\Omega} = 0$ Korn's inequality [3, Chapter III, §3.2] $\|\vec{u}\|_{1,p,\Omega} \leq C_k(\Omega)\|\mathbb{D}(\vec{u})\|_{p,\Omega}$, $p > 1$ is valid. This fact allows to introduce the special norm $\|\cdot\|$ in V by means of the equality

$$\|\vec{v}\| = (\int_\Omega |\mathbb{D}(\vec{v})|^p dx)^{1/p}.$$

Using this definition rewrite (2.10) in the form

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}_m(t)\|^2_{2,\Omega} + \tilde{m}\|\vec{u}_m\|^p \leq C_k(\Omega)\|\vec{f}(t)\|_{V'}\|\vec{u}_m\|. \tag{2.11}$$

Integrating with respect to t over $(0,t)$ we obtain

$$\frac{\|\vec{u}_m(t)\|^2_{2,\Omega}}{2} + \tilde{m}\int_0^t \|\vec{u}_m(s)\|^p ds \leq C_k(\Omega)\int_0^t \|\vec{f}(s)\|_{V'}\|\vec{u}_m(s)\|ds + \frac{\|\vec{u}_{0m}\|^2_{2,\Omega}}{2}. \tag{2.12}$$

Using simple arguments we deduce from this inequality the following a priori estimates.

$$\|\vec{u}_m\|_{L_p(0,T;V)} \leq C_k(\Omega)\max\left\{\left(\frac{\|\vec{f}\|_{L_{p'}(0,T;V')}}{\tilde{m}}\right)^{1/(p-1)}, \left(\frac{\|\vec{u}_{0m}\|^2_{2,\Omega}}{2\tilde{m}}\right)^{1/p}\right\} \equiv C_{1m} \leq C_1, \tag{2.13}$$

$$\|\vec{u}_m\|_{L_\infty(0,T;H_0(\Omega))} \leq 2\|\vec{f}\|_{L_{p'}(0,T;V')}\max\left\{\left(\frac{C_k(\Omega)}{\tilde{m}}\|\vec{f}\|_{L_{p'}(0,T;V')}\right)^{1/(p-1)}, \left(\frac{\|\vec{u}_{0m}\|^2_{2,\Omega}}{2\tilde{m}}\right)^{1/p}\right\} \equiv C_{2m} \leq C_2, \tag{2.14}$$

where C_1, C_2 do not depend on m. The following a priori estimate for $\vec{u}'_m$ is obtained by utilizing the same arguments as in [6, Chapter II, §5.2].

$$\|\vec{u}'_m\|_{L^{p'}(0,T;V^{-3}(\Omega))} \leq C_3, \quad C_3 \text{ does not depend on } m. \tag{2.15}$$

Note that this estimate is not necessarily uniform with respect to ε.

For justification of solvability of Galerkin system, let us utilize Shauder principle of a fixed point. We introduce the relevant completely continuous operator as follows.

Let $Z = \{\vec{\psi}(t)\,|\,\vec{\psi}(t) \in C([0,T]),\ \|\vec{\psi}\|_{C([0,T])} \leq C_{2m},\ \vec{\psi} = (c_{m1}(t), \ldots, c_{mm}(t)),\ c_{mi}(t) = c_{mi},\ i = 1, \ldots, m\}$. Clearly, Z is the bounded convex closed set in $C([0,T])$. Let $\vec{\psi}^0(t) = (c^0_{m1}(t), \ldots, c^0_{mm}(t))$ is an element from Z. Construct the vector $\vec{u} = \sum_{k=1}^m c^0_{mk}(t)\vec{w}_k(x)$. Evidently, $\vec{u}(t) \in C([0,T]; C^1(\Omega))$. For the given $\vec{u}(t)$ find $\tilde{\mu}$ which is solution to the Cauchy problem for transport equation

$$D_t\tilde{\mu} + \vec{u}\nabla\tilde{\mu} = 0, \quad \tilde{\mu}|_{t=0} = \mu_{0m}. \tag{2.16}$$

Such solution exists, is unique and belongs to $C^1(Q_T)$ [5].

Next, find a solution $\vec{u}_1 = \sum_{k=1}^m c^1_{mk}(t)\vec{w}_k(x)$ to the system of ODEs

$$\int_\Omega \vec{u}'_1\vec{w}_j dx + \int_\Omega \vec{u}(t)\otimes\vec{w}_j : \nabla\vec{u}_1(t)dx + \int_\Omega \vec{a}(\tilde{\mu}, \vec{u}_1(t))\vec{w}_j dx + \frac{1}{\varepsilon}\int_\Omega \beta(\vec{u}_1(t))\vec{w}_j dx = \int_\Omega \vec{f}(t)\vec{w}_j dx. \tag{2.17}$$

Repeating the considerations from [7, Chapter 3, §1.2] and basing on the estimate (2.14) we conclude that the mapping $A:\ \vec{\psi}_1(t) = A[\vec{\psi}_0](t)$, $\vec{\psi}_1(t) = \{c^1_{m1}(t), \ldots, c^1_{mm}(t)\}$, constructed by virtue of formulae (2.17), (2.16) appears to be the completely continuous operator that self-maps Z in the norm of $C([0,T])$. Thus, there exists a fixed point $\vec{\psi}$ in Z. Hence the system (2.4)–(2.8) has a solution for any m.

Our next goal is to fulfil limiting transition in Galerkin system as $m \to \infty$. For this purpose we make use of the compactness theorem form [6, Chapter 1, Theorem 5.1]. Since imbedding $V \to H(\Omega)$ is compact due to the estimates (2.13)–(2.15) one can extract a subsequence $\vec{u}_\nu$ such that

$$\begin{aligned} \vec{u}_\nu \to \vec{u} \qquad & \text{weakly in } L_p(0,T;V), \text{ weak-star in } L_\infty(0,T;H(\Omega)), \\ & \text{in } L_p(0,T;H(\Omega)), \text{ a. e. in } Q_T, \\ D_t\vec{u}_\nu \to D_t\vec{u} \qquad & \text{weakly in } L_{p'}(0,T;V^{-3}(\Omega)), \\ \frac{1}{\varepsilon}\beta(\vec{u}_\nu) + \vec{a}(\mu_\nu, \vec{u}_\nu) \to \chi \qquad & \text{weakly in } L_{p'}(0,T;V'). \end{aligned} \tag{2.18}$$

Due to (2.18), (2.7) and convergence theorem for transport equations [8] we deduce

$$\mu_\nu \to \mu \text{ in } C([0,T];L_q(\Omega)), \quad 1 \le q < \infty. \tag{2.19}$$

Repeating the arguments from [6, Chapter II, §5.2] we obtain

$$\int_\Omega (\vec{u}'\vec{v} + \chi\vec{v} + \vec{u}\otimes\vec{v} : \nabla\vec{u})dx = \int_\Omega \vec{f}\vec{v}dx \quad \forall\, \vec{v} \in V. \tag{2.20}$$

Observe that $\vec{a}$ is the gradient of the convex functional

$$\Psi : \vec{u} \to \frac{1}{p}\int_\Omega \mu|\mathbb{D}(\vec{u})|^p dx.$$

Hence, $\vec{a}(\mu,\vec{u})$ is monotonuous function of $\vec{u}$. Utilizing this fact, formula (2.19) and working out the considerations analogous to those from [6, Chapter II, §5.1] we get

$$\chi = \vec{a}(\mu,\vec{u}) + \frac{1}{\varepsilon}\beta(\vec{u}). \tag{2.21}$$

This identity together with (2.20) yields the assertion of Lemma 2.1. □

2.2 Passage to limit as $\varepsilon \to 0$

Denote by $\{\vec{u}_\varepsilon, \mu_\varepsilon\}$ a generalized solution of Problem A_ε that corresponds to a fixed value of parameter ε. Let us obtain uniform with respect to ε a priori estimates on solutions of Problem A_ε.

2.2.1 A priori estimates

Substituting $\vec{v} = \vec{u}_\varepsilon$ into (2.2) and integrating over $(0,t)$ with respect to t we deduce

$$\begin{aligned} \frac{\|\vec{u}_\varepsilon(t)\|^2_{2,\Omega}}{2} + \int_0^t\int_\Omega \mu_\varepsilon|\mathbb{D}(\vec{u}_\varepsilon)|^p dxds + \frac{1}{\varepsilon}\int_0^t\int_\Omega \beta(\vec{u}_\varepsilon)\vec{u}_\varepsilon dxds & \\ \le C_k(\Omega)\|\vec{f}\|_{L_{p'}(0,T;V')}\left(\int_0^t \|\vec{u}_\varepsilon(s)\|^p ds\right)^{1/p} + \frac{\|\vec{u}_0\|^2_{2,\Omega}}{2}. & \end{aligned} \tag{2.22}$$

Hence,

$$\begin{aligned} \frac{\|\vec{u}_\varepsilon(t)\|^2_{2,\Omega}}{2} + \tilde{m}\int_0^t \|\vec{u}_\varepsilon(s)\|^p ds + \frac{1}{\varepsilon}\int_0^t\int_\Omega \beta(\vec{u}_\varepsilon)\vec{u}_\varepsilon dxds & \\ \le C_k(\Omega)\|\vec{f}\|_{L_{p'}(0,T;V')}\left(\int_0^t \|\vec{u}_\varepsilon(s)\|^p ds\right)^{1/p} + \frac{\|\vec{u}_0\|^2_{2,\Omega}}{2}. & \end{aligned}$$

This yields

$$\left(\int_0^t \|\vec{u}_\varepsilon(s)\|^p ds\right)^{1/p} \le \max\left\{\left(\frac{C_k(\Omega)}{\tilde{m}}\|\vec{f}\|_{L_{p'}(0,T;V')}\right)^{1/(p-1)}, \left(\frac{\|\vec{u}_0\|^2_{2,\Omega}}{2\tilde{m}}\right)^{1/p}\right\} \equiv C_4. \tag{2.23}$$

From (2.23) using Korn's inequality we deduce a priori estimates

$$\|\vec{u}_\varepsilon\|_{L_p(0,T;V)} \le C_k(\Omega)\max\left\{\left(\frac{\|\vec{f}\|_{L_{p'}(0,T;V')}}{\tilde{m}}\right)^{1/(p-1)}, \left(\frac{\|\vec{u}_0\|^2_{2,\Omega}}{2\tilde{m}}\right)^{1/p}\right\} \equiv C_5, \tag{2.24}$$

$$\|\vec{u}_\varepsilon\|_{L^\infty(0,T;H_0(\Omega))} \le 2\|\vec{f}\|_{L_{p'}(0,T;V')}C_4 \equiv C_6, \tag{2.25}$$

$$0 \le \tfrac{1}{\varepsilon}\textstyle\int_0^T\int_\Omega \beta(\vec{u}_\varepsilon)\vec{u}_\varepsilon dxds \le C_k(\Omega)\|\vec{f}\|_{L^{p'}(0,T;V')}C_4 + \|\vec{u}_0\|^2_{2,\Omega} \equiv C_7. \tag{2.26}$$

Using Hölder's inequality and formula (2.23) we obtain

$$\|\vec{a}(\mu_\varepsilon, \vec{u}_\varepsilon)\|_{L_{p'}(0,T;V')} \le C_4^{p-1}\tilde{M}. \tag{2.27}$$

2.3 A priori estimate for $D_t\vec{u}_\varepsilon$

First, using a priori estimates (2.23)–(2.26) we evaluate the norm of the functional

$$\vec{v} \to \frac{1}{\varepsilon}\int_{Q_T} \beta(\vec{u}_\varepsilon)\vec{v}dxdt.$$

Consider

$$I_\varepsilon = \frac{1}{\varepsilon}\int_\Omega \beta(\vec{u}_\varepsilon)\vec{v}dx = \frac{1}{\varepsilon}\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{\frac{p}{2}-1}\mathbb{D}(\vec{u}_\varepsilon) : \mathbb{D}(\vec{v})dx,$$

$$|I_\varepsilon| \le \max_\Omega |\mathbb{D}(\vec{v})|\Big(\frac{1}{\varepsilon}\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{\frac{p}{2}-1}|\mathbb{D}(\vec{u}_\varepsilon)|dx\Big)$$
$$\le C_s^{(1)}(\Omega)\|\vec{v}\|_{H^3(\Omega)}\Big(\frac{1}{\varepsilon}\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{\frac{p}{2}-1}|\mathbb{D}(\vec{u}_\varepsilon)|dx\Big).$$

The latter inequality is valid since

$$\max_\Omega |\mathbb{D}(\vec{v})| \le C_s^{(1)}(\Omega)\|\mathbb{D}(\vec{v})\|_{W_2^2(\Omega)}, \quad C_s^{(1)}(\Omega)\|\mathbb{D}(\vec{v})\|_{W_2^2(\Omega)} \le C_s^{(1)}(\Omega)\|\vec{v}\|_{H^3(\Omega)},$$

due to Sobolev embedding theorem. Here, $C_s^{(1)}$ is the constant in Sobolev embedding theorem.

Applying Hölder's inequality twice, we deduce

$$\int_0^T |I_\varepsilon(t)|dt \le C_s^{(1)}(\Omega)\|\vec{v}\|_{L_s(0,T;H^3(\Omega))}\frac{1}{\varepsilon}\Big(\int_0^T \Big\{\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{\frac{p}{2}-1}|\mathbb{D}(\vec{u}_\varepsilon)|dx\Big\}^r dt\Big)^{1/r},$$

where $s^{-1} + r^{-1} = 1$, $1 < s, r < \infty$), and

$$\Big\{\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{\frac{p}{2}-1}|\mathbb{D}(\vec{u}_\varepsilon)|dx\Big\}^r \le (\mathrm{meas}\,\Omega)^{r/s}\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{r(\frac{p}{2}-1)}|\mathbb{D}(\vec{u}_\varepsilon)|^r dx$$
$$= (\mathrm{meas}\,\Omega)^{r/s}\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{\frac{p}{2}-1}([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{(r-1)(\frac{p}{2}-1)}|\mathbb{D}(\vec{u}_\varepsilon)|^r dx \le \dots,$$

let us make use of the simple bound $[|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+ \le |\mathbb{D}(\vec{u}_\varepsilon)|^2$,

$$\dots \le (\mathrm{meas}\,\Omega)^{r/s}\int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{\frac{p}{2}-1}|\mathbb{D}(\vec{u}_\varepsilon)|^{2(r-1)(\frac{p}{2}-1)+r}dx.$$

Choosing r and s satisfying $r = p(p-1)^{-1}$, which yields $s = p$, we obtain in view of previous investigations that

$$\int_0^T |I_\varepsilon(t)|dt \le C_s^{(1)}(\Omega)\|\vec{v}\|_{L_p(0,T;V^3(\Omega))}(\mathrm{meas}\,\Omega)^{1/(p-1)}\frac{1}{\varepsilon}\int_0^T\int_\Omega \beta(\vec{u}_\varepsilon)\vec{u}_\varepsilon dxds. \tag{2.28}$$

Finally, formulae (2.26), (2.28) yield

$$\|\frac{1}{\varepsilon}\beta(\vec{u}_\varepsilon)\|_{L_{p'}(0,T;V^{-3}(\Omega))} \le C_s^{(1)}(\Omega)(\mathrm{meas}\,\Omega)^{1/(p-1)}C_7 \equiv C_8. \tag{2.29}$$

Let us establish other two necessary bounds. Observe that

$$\Big|\int_{Q_T} \vec{a}(\mu_\varepsilon, \vec{u}_\varepsilon)\vec{v}dxdt\Big| \le \Big(\int_0^T \Big(\int_\Omega \mu_\varepsilon|\mathbb{D}(\vec{u}_\varepsilon)|^{p-1}dx\Big)^{p'}dt\Big)^{1/p'}\|\nabla\vec{v}\|_{L_p(0,T;C(\Omega))}$$
$$\le \|\mu_\varepsilon\|_{C([0,T];L_p(\Omega))}^{1/p'}\|\mathbb{D}(\vec{u}_\varepsilon)\|^{1/(p-1)}\|\nabla\vec{v}\|_{L_p(0,T;C(\Omega))}. \tag{2.30}$$

Due to Sobolev embedding theorem $\|\vec{v}\|_{L_p(0,T;C(\Omega))} \le C_s^{(1)}(\Omega)\|\vec{v}\|_{L_p(0,T;V^3(\Omega))}$. Hence, using formula (2.23) we deduce from (2.30) the following.

$$\Big|\int_{Q_T} \vec{a}(\mu_\varepsilon, \vec{u}_\varepsilon)\vec{v}dxdt\Big| \le C_s^{(1)}(\Omega)\|\mu_\varepsilon\|_{C([0,T];L_p(\Omega))}^{1/p'}C_4^{p'}\|\vec{v}\|_{L_p(0,T;V^3(\Omega))}. \tag{2.31}$$

Since $\|\mu_\varepsilon\|_{C([0,T];L_p(\Omega))} = \|\mu_0\|_{L_p(\Omega)}$, $\|\mu_0\|_{L_p(\Omega)} \le \tilde{M}(\text{meas}\,\Omega)^{1/p}$ [7, Chapter 3, §2, Lemma 2.1] we have

$$\|\vec{a}(\mu_\varepsilon, \vec{u}_\varepsilon)\|_{L_{p'}(0,T;V^{-3}(\Omega))} \le C_s^{(1)}(\Omega,p)\tilde{M}^{1/p'}(\text{meas}\,\Omega)^{1/p\,p'}C_4^{p'} \equiv C_9. \tag{2.32}$$

Next,

$$\left|\int_{Q_T} \vec{u}_\varepsilon \otimes \vec{v} : \nabla\vec{u}_\varepsilon dxdt\right| = \left|\int_{Q_T} \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon : \nabla\vec{v} dxdt\right| \le \left(\int_0^T \max_\Omega |\nabla\vec{v}|dt\right)\|\vec{u}_\varepsilon\|^2_{L_\infty(0,T;H(\Omega))}$$
$$\le C_s^{(1)}(\Omega)\|\vec{v}\|_{L_p(0,T;V^3(\Omega))}(\text{meas}\,\Omega)^{1/p'}\|\vec{u}_\varepsilon\|^2_{L_\infty(0,T;H(\Omega))}. \tag{2.33}$$

Let $g_\varepsilon = \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon$. In strength of the inequalities (2.25), (2.34) we obtain

$$\|g_\varepsilon\|_{L_{p'}(0,T;V^{-3}(\Omega))} \le C_s^{(1)}(\Omega)(\text{meas}\,\Omega)^{1/p'}C_6^2 \equiv C_{10}. \tag{2.34}$$

Using the bounds (2.28), (2.32) and (2.34) and taking into account that $\vec{f} \in L_{p'}(0,T;V')$ in strength of formula (2.2) we establish the bound on $D_t\vec{u}_\varepsilon$:

$$\|D_t\vec{u}_\varepsilon\|_{L_{p'}(0,T;V^{-3}(\Omega))} \le C_8 + C_9 + C_{10} + \|\vec{f}\|_{L_{p'}(0,T;V')}C_s^{(2)}(\Omega,p) \equiv C_{11}, \tag{2.35}$$

where $C_s^{(2)}(\Omega,p)$ is the constant in the inequality in Sobolev embedding theorem:
$\|\vec{v}\|_{L_p(0,T;V)} \le C_s^{(2)}(\Omega,p)\|\vec{v}\|_{L_p(0,T;V^3(\Omega))}$.

2.4 Passage to limit as $\varepsilon \to 0$

2.4.1

Due to a priori estimates (2.24)–(2.27) and (2.35) one can extract a subsequence $\{\vec{u}_\varepsilon, \mu_\varepsilon\}$ such that

$$\vec{u}_\varepsilon \to \vec{u} \quad \text{weakly in } L_p(0,T;V), \quad \text{weak-star in } L_\infty(0,T;H(\Omega)), \tag{2.36}$$
$$\vec{u}'_\varepsilon \to \vec{u}' \quad \text{weakly in } L_{p'}(0,T;V^{-3}(\Omega)), \tag{2.37}$$
$$\vec{a}(\mu_\varepsilon, \vec{u}_\varepsilon) \to \chi_* \quad \text{weakly in } L_{p'}(0,T;V'). \tag{2.38}$$

Using the compactness theorem [6, Chapter I, §5, Theorem 5.1] in strength of (2.36) and (2.37) we conclude that

$$\vec{u}_\varepsilon \to \vec{u} \text{ in } L_p(0,T;H(\Omega)) \text{ and a. e. in } Q_T. \tag{2.39}$$

From this limiting relation and convergence theorem for transport equations [8] we obtain

$$\mu_\varepsilon \to \mu \text{ in } C([0,T];L_\vartheta(\Omega)). \tag{2.40}$$

Besides, since $\tilde{m} \le \mu_\varepsilon(x,t) \le \tilde{M}$ a. e. in Q_T the bound $\tilde{m} \le \mu(x,t) \le \tilde{M}$ is valid a. e. in Q_T.

In strength of the inequality

$$0 \le \int_\Omega ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{p/2}dx \le \int_\Omega \beta(\vec{u}_\varepsilon)\vec{u}_\varepsilon dx$$

and the estimate (2.26) we obtain that

$$\int_{Q_T} ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{p/2}dxdt \le C_7\varepsilon \to 0 \tag{2.41}$$

as $\varepsilon \to 0$. Hence

$$|\mathbb{D}(\vec{u})(x,t)| \le M \text{ a. e. in } Q_T. \tag{2.42}$$

Passing to limit as $\varepsilon \to 0$ in the integral identity

$$\int_{Q_T} \mu_\varepsilon(D_t\psi + \vec{u}_\varepsilon\nabla\psi)dxdt + \int_\Omega \mu_0\psi(x,0)dx = 0$$

due to (2.36) and (2.40) we conclude that the integral identity (1.8) holds.

2.4.2 Passage to limit as $\varepsilon \to 0$ (completion). Utilization of monotonity method

Consider

$$X_\varepsilon = \int_{Q_T} [D_t\vec{\varphi}(\vec{\varphi}-\vec{u}_\varepsilon) + \vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)(\vec{\varphi}-\vec{u}_\varepsilon) + \vec{u}_\varepsilon \otimes (\vec{\varphi}-\vec{u}_\varepsilon) : \nabla\vec{u}_\varepsilon - \vec{f}(\vec{\varphi}-\vec{u}_\varepsilon)]dxdt, \tag{2.43}$$

where $\vec{\varphi} \in W_p^1(0,T;V)$, $|\mathbb{D}(\vec{\varphi})| \le M$ a. e. in Q_T, and

$$\text{either} \quad \vec{\varphi}(0) = \vec{u}_0 \quad \text{or} \quad \|\vec{\varphi}(T)\|^2_{2,\Omega} \ge C_6^2 - \|\vec{\varphi}(0) \pm \vec{u}_0\|^2_{2,\Omega}. \tag{2.44}$$

Due to identity (2.2) (note that $\beta(\vec{\varphi}) = 0$) since operator $\vec{v} \to \beta(\vec{v})$ is monotonous the equality

$$X_\varepsilon = \int_{Q_T} [(D_t\vec{\varphi} - D_t\vec{u}_\varepsilon)(\vec{\varphi}-\vec{u}_\varepsilon) + \frac{1}{\varepsilon}(\beta(\vec{\varphi}) - \beta(\vec{u}_\varepsilon))(\vec{\varphi}-\vec{u}_\varepsilon)]dxdt \ge 0 \tag{2.45}$$

is valid. Combining (2.43) and (2.45) we obtain

$$\int_{Q_T} [D_t\vec{\varphi}(\vec{\varphi}-\vec{u}_\varepsilon) + \vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{\varphi} + \vec{u}_\varepsilon \otimes (\vec{\varphi}-\vec{u}_\varepsilon) : \nabla\vec{u}_\varepsilon - \vec{f}(\vec{\varphi}-\vec{u}_\varepsilon)]dxdt \ge \int_{Q_T} \vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{u}_\varepsilon dxdt. \tag{2.46}$$

In strength of (2.36), (2.38) and (2.39), passing to limit as $\varepsilon \to 0$ in the inequality (2.46) we deduce

$$\int_{Q_T} [D_t\vec{\varphi}(\vec{\varphi}-\vec{u}) + \chi_*\vec{\varphi} + \vec{u} \otimes (\vec{\varphi}-\vec{u}) : \nabla\vec{u} - \vec{f}(\vec{\varphi}-\vec{u})]dxdt \ge \liminf_{\varepsilon\to 0} \int_{Q_T} \vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{u}_\varepsilon dxdt. \tag{2.47}$$

Proposition 2.1. *The following inequality is valid.*

$$\liminf_{\varepsilon\to 0} \int_{Q_T} \vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{u}_\varepsilon dxdt \ge \int_{Q_T} \vec{a}(\mu,\vec{u})\vec{u} dxdt.$$

The Verification of Proposition 2.1 is based on the following well-known statement [9, Chapter 5].

Proposition 2.2. *Let X be reflexive Banach space, $\vec{v}_\varepsilon \to \vec{v}$ weakly in X. Then the following inequality is valid.*

$$\liminf_{\varepsilon\to 0} \|\vec{v}_\varepsilon\|_X \ge \|\vec{v}\|_X.$$

Proof of Proposition 2.1.

$$\int_{Q_T} \vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{u}_\varepsilon dxdt = \int_{Q_T} \mu_\varepsilon|\mathbb{D}(\vec{u}_\varepsilon)|^p dxdt = \int_{Q_T} (\mu_\varepsilon - \mu)|\mathbb{D}(\vec{u}_\varepsilon)|^p dxdt + \int_{Q_T} \mu|\mathbb{D}(\vec{u}_\varepsilon)|^p dxdt \equiv I_1^\varepsilon + I_2^\varepsilon.$$

Here,

$$|I_1^\varepsilon| \le \int_{Q_T} |\mu_\varepsilon - \mu|(|\mathbb{D}(\vec{u}_\varepsilon)|^p - M^p)dxdt + \int_{Q_T} |\mu_\varepsilon - \mu|M^p dxdt = I_{11}^\varepsilon + I_{12}^\varepsilon.$$

$I_{12}^\varepsilon \to 0$ in strength of (2.40), $I_{11}^\varepsilon \to 0$ since $I_{11}^\varepsilon \ge -\int_{Q_T}|\mu_\varepsilon - \mu|M^p dxdt \to 0$ as $\varepsilon \to 0$, and

$$\begin{aligned} I_{11}^\varepsilon \le \int_{Q_T} |\mu_\varepsilon - \mu|[|\mathbb{D}(\vec{u}_\varepsilon)|^p - M^p]^+ dxdt &\le \int_{Q_T} |\mu_\varepsilon - \mu|([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{p/2} dxdt \\ &\le (\tilde{M} - \tilde{m}) \int_{Q_T} ([|\mathbb{D}(\vec{u}_\varepsilon)|^2 - M^2]^+)^{p/2} dxdt \le (\tilde{M} - \tilde{m})C_7\varepsilon \to 0 \end{aligned}$$

as $\varepsilon \to 0$ in strength of (2.41). Thus,

$$I_1^\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{2.48}$$

Next, $I_2^\varepsilon = \int_{Q_T} |\mu^{1/p}\mathbb{D}(\vec{u}_\varepsilon)|^p dxdt$. Observe that functional

$$\vec{v} = \{v_i\}_{i=1}^9 \to \|\vec{v}\|_{(1)} \equiv \left[\int_{Q_T} (\sum_{i=1}^9 v_i^2)^{p/2} dxdt\right]^{1/p}$$

may be introduced as a norm in $(L_p(Q_T))^9$ and that $\mu^{1/p}\mathbb{D}(\vec{u}_\varepsilon) \to \mu^{1/p}\mathbb{D}(\vec{u})$ weakly in $(L_p(Q_T))^9$ since the norm $\|\mu^{1/p}\mathbb{D}(\vec{u}_\varepsilon)\|_{(1)}$ is uniformly with respect to ε bounded, and

$$\int_{Q_T} \mu^{1/p} D_{ij}(\vec{u}_\varepsilon)dxdt \to \int_{Q_T} \mu^{1/p}\mathbb{D}(\vec{u}_\varepsilon)dxdt, \; i,j = 1,2,3,$$

since $\mu^{1/p} \in L_{p'}(Q_T)$ and $\vec{u}_\varepsilon \to \vec{u}$ weakly in $L_p(0,T;V)$.

Hence, in view of Proposition 2.2

$$\liminf_{\varepsilon\to 0} I_2^\varepsilon \equiv \liminf_{\varepsilon\to 0} \|\mu^{1/p}\mathbb{D}(\vec{u}_\varepsilon)\|_{(1)} \geq \|\mu^{1/p}\mathbb{D}(\vec{u})\|_{(1)} \equiv \int_{Q_T} \vec{a}(\mu,\vec{u})\vec{u}dxdt. \tag{2.49}$$

□

Thus, from (2.47) we conclude that

$$\int_{Q_T} [D_t\vec{\varphi}(\vec{\varphi}-\vec{u}) + \chi_*\vec{\varphi} + \vec{u}\otimes(\vec{\varphi}-\vec{u}) : \nabla\vec{u} - \vec{f}(\vec{\varphi}-\vec{u})]dxdt \geq \int_{Q_T} \vec{a}(\mu,\vec{u})\vec{u}dxdt. \tag{2.50}$$

Now, let us prove that

$$\chi_* = \vec{a}(\mu,\vec{u}). \tag{2.51}$$

Observe

$$Y_\varepsilon = \int_{Q_T} \mu(|\mathbb{D}(\vec{u}_\varepsilon)|^{p-2}\mathbb{D}(\vec{u}_\varepsilon) - |\mathbb{D}(\vec{\gamma})|^{p-2}\mathbb{D}(\vec{\gamma})) : \mathbb{D}(\vec{u}_\varepsilon - \vec{\gamma})dxdt, \tag{2.52}$$

where $\vec{\gamma}$ is an arbitrary smooth function. Due to monotonity of the operator $\vec{v} \to \vec{a}(\mu,\vec{v})$ we deduce from (2.52) that

$$\liminf Y_\varepsilon \geq 0. \tag{2.53}$$

Combining (2.46), (2.52) and (2.53) and passing to limit as $\varepsilon \to 0$ we arrive at

$$\int_{Q_T} D_t\vec{\varphi}(\vec{\varphi}-\vec{u})dxdt + \int_{Q_T} \chi_*\vec{\varphi}dxdt - \int_{Q_T} \vec{u}\otimes\vec{u} : \nabla\vec{\varphi}dxdt - \int_{Q_T} \vec{f}(\vec{\varphi}-\vec{u})dxdt$$
$$- \int_{Q_T} \chi_*\vec{\gamma}dxdt - \int_{Q_T} \mu|\mathbb{D}(\vec{\gamma})|^{p-2}\mathbb{D}(\vec{\gamma}) : \mathbb{D}(\vec{u}-\vec{\gamma})dxdt \geq \liminf Y_\varepsilon \geq 0. \tag{2.54}$$

Substitute a test function $-\vec{\varphi}$ into the inequality (2.50) on the place of $\vec{\varphi}$ (what is legal in strength of condition (2.44)):

$$-\int_{Q_T} D_t\vec{\varphi}(\vec{\varphi}+\vec{u})dxdt - \int_{Q_T} \chi_*(\vec{\varphi}+\vec{u})dxdt - \int_{Q_T} \mu|\mathbb{D}(\vec{u})|^p dxdt + \int_{Q_T} \vec{u}\otimes\vec{u} : \nabla\vec{\varphi}dxdt$$
$$\geq -\int_{Q_T} \vec{f}(\vec{\varphi}+\vec{u})dxdt.$$

This inequality together with the formula (2.54) gives

$$-\int_{Q_T} \chi_*\vec{\gamma}dxdt - \int_{Q_T} \mu|\mathbb{D}(\vec{\gamma})|^{p-2}\mathbb{D}(\vec{\gamma}) : \mathbb{D}(\vec{u}-\vec{\gamma})dxdt$$
$$\geq -2\int_{Q_T} \vec{f}\vec{u}dxdt + 2\int_{Q_T} D_t\vec{\varphi}\vec{u}dxdt + \int_{Q_T} \mu|\mathbb{D}(\vec{u})|^p dxdt. \tag{2.55}$$

Substitute as a test function into (2.50) the following expression: $\vec{\varphi} = ((\vartheta_m\vec{u}) * \rho_n * \rho_n)\vartheta_m$, $n > 2m$. Here ϑ_m, ρ_n are regularizing sequences defined as follows: ϑ_m is continuous piecewise linear function on $[0,T]$, $\vartheta_m(t) = 1$ if $0 + \frac{2}{m} < t < T - \frac{2}{m}$, $\vartheta_m(t) = 0$ if $t > T - \frac{1}{m}$ and $t < \frac{1}{m}$; ρ_m is regularizing sequence in $\mathcal{D}(\mathbb{R})$, $\rho_n(t) = \rho_n(-t)$, $\int_{\mathbb{R}} \rho_n(t)dt = 1$, supp$\rho_n \subset [-n^{-1}, n]$.

Passing firstly to the limit as $n \to \infty$, then as $m \to \infty$, and repeating the considerations from [6, Chapter 2, §5.2] we establish that $\int_{Q_T} \chi_*\vec{u}dxdt \geq \int_{Q_T} \mu|\mathbb{D}(\vec{u})|^p dxdt \geq 0$. Hence, $-\int_{Q_T} \chi_*\vec{u}dxdt \leq \int_{Q_T} \mu|\mathbb{D}(\vec{u})|^p dxdt$. Formula (2.55) and this inequality lead to

$$\int_{Q_T} (\chi_* - \vec{a}(\mu,\vec{\gamma}))(\vec{u}-\vec{\gamma})dxdt \geq -2\int_{Q_T} \vec{f}\vec{u}dxdt + 2\int_{Q_T} \vec{\varphi}_t\vec{u}dxdt.$$

Imposing $\vec{\varphi}_t = \vec{f}$ (this is legal due to conditions in the statement of Theorem 2.1) we have

$$\int_{Q_T} (\chi_* - \vec{a}(\mu,\vec{\gamma}))(\vec{u}-\vec{\gamma})dxdt \geq 0 \quad \forall\vec{\gamma} \in L_p(0,T;V).$$

By standard arguments [6], this yields the formula (2.51).

Thus, the formulae (2.42), (2.44), (2.50) and (2.51) show that the integral identity (1.7) holds. Hence, the pair of functions $\{\vec{u},\mu\}$ form a generalized solution of Problem A. Theorem 2.1 is proved.

3 Solvability Of Problem On Motion Of Solid Bodies In Non-Newtonian fluid

We prove the following existence theorem.

Theorem 3.1. *Let* $\vec{u}_0 \in H_0(\Omega) \cap K(\Lambda_0)$, $|\mathbb{D}(\vec{u}_0)| \le M$ *a. e. in* Q_T,

$$\vec{f} \in L^{p'}(0,T;V'), \quad |\mathbb{D}(\vec{F})| \le M, \text{ where } D_t\vec{F} = \vec{f}, \quad p^{-1} + (p')^{-1} = 1, \quad p \ge \frac{11}{5}. \tag{3.1}$$

Then there exists a generalized solution to Problem B.

As it has already been noticed in Section 1 of the paper, the main idea of justification of Theorem 3.1 consists of utilization of *solidification method*.

3.1 Additional regularity properties of solutions of Problem A

In order to prove Theorem 3.1, we will use some properties of solutions of Problem A that are not involved in Definition 1.1.

Lemma 3.1. . *Let* $\vec{u} \in \{\vec{v} \,|\, \vec{v} : \Omega \times [0,T] \to \mathbb{R}^3, \quad |\mathbb{D}(\vec{v})(x,t)| \le M \text{ a. e. in } Q_T\}$. *Then the estimate*

$$|\vec{u}(x') - \vec{u}(x'')| \le C_{12} M |x' - x''|(\ln|x' - x''| + 1), \; C_{12} = C_{12}(\Omega)$$

is valid for any rather close to each other point x', $x'' \in \Omega$.

Proof. Let us utilize the integral representation of the components of velocity vector in terms of the components of deformation tensor [3, Chapter III, §2]:

$$u_i(x') - u_i(x'') = \int_\Omega \sum_{k,l=1}^{3} [R_i^{kl}(x',y) - R_i^{kl}(x'',y)] D_{kl}(\vec{u})(y)dy.$$

Here, $R_i^{kl}(x,y) = (\omega_{kl}^i(x,y)/|x-y|^{n-1}) - \theta_{kl}^i(x,y)$, where $\omega_{kl}^i(x,y)$ are in $C^\infty(R^n \setminus \{x\})$, and are positively homogeneous functions of zero order with respect to $x-y$; $\theta_{kl}^i \in C^\infty(R^n \times R^n)$. The condition in statement of the lemma follows to the estimate

$$|\vec{u}(x') - \vec{u}(x'')| \le M \int_\Omega |R(x',y) - R(x'',y)|dy.$$

Introduce $B = \{y| \;\; |y-x| \le 2d, \; d = |x'-x''|\}$. The assertion of Lemma 3.1 is immediately achieved in strength of the bounds

$$I_1 = \int_B |R(x',y) - R(x'',y)|dy \le C_{13}\, d, \quad \int_{\Omega\setminus B} |R(x',y) - R(x'',y)|dy \le C_{14}\, d(1 + \ln|d|).$$

□

Lemma 3.2. *Let function* $\vec{u} : \Omega \times [0,T] \to \mathbb{R}^3$ *be such that* $|\mathbb{D}(\vec{u})(x,t)| \le M$ *for a. e.* $(x,t) \in Q_T$, $\vec{u}|_{\partial\Omega} = 0$ *for a. e.* $t \in [0,T]$. *Then* $\vec{u} \in L_\infty(0,T;C(\Omega)) \cap L_\infty(0,T;W^1_\vartheta(\Omega))$ $\forall \epsilon < \infty$, *and*

$$\max_\Omega |\vec{u}| \le C_s^{(3)}(\Omega) C_k(\Omega,1,\epsilon) M \equiv C_{15} \text{ for a. e. } t \in [0,T], \tag{3.2}$$

$$\|\vec{u}\|_{L_\infty(0,T;W^1_\epsilon(\Omega))} \le C_k(\Omega,1,\epsilon) M\, T^{1/\epsilon} \equiv C_{16}(\epsilon). \tag{3.3}$$

Proof. In strength of Korn's inequality and the boundary condition $\vec{u}|_{\partial\Omega} = 0$ the bound

$$\|\vec{u}\|_{W^1_\epsilon(\Omega)} \le C_k(\Omega,1,\epsilon)\|\mathbb{D}(\vec{u})\|_{\epsilon,\Omega} \le C_k(\Omega,1,\epsilon) M$$

is valid for a. e. $t \in [0,T]$. Due to Sobolev embedding theorem

$$\|\vec{u}\|_{C(\Omega)} \le C_s^{(3)}(\Omega)\|\vec{u}\|_{W^1_\epsilon(\Omega)} \text{ for a. e. } t \in [0,T], \; \epsilon > 3.$$

□

3.2 Proof of Theorem 3.1

3.2.1 Choice of initial data

We consider Problem A provided with initial data of the following form.

$$\mu_\varepsilon(x,0) = \begin{cases} 1, & x \in \Omega \setminus V_0, \\ 1/\varepsilon, & x \in V_0, \end{cases} \tag{3.4}$$

and function $\vec{u}_\varepsilon(x,0) = \vec{u}_0(x)$ satisfies the conditions of Theorem 3.1.

Due to Theorem 2.1 there exists a generalized solution of Problem A provided with such initial data. Denote this solution by $\{\vec{u}_\varepsilon(x,t),\, \mu_\varepsilon(x,t)\}$.

3.2.2 Shift operator

In order to study the evolution of initial data (3.4) in time and space, introduce the *shift operator* $\Phi_\varepsilon^{t_1,t_2}$ as follows. Consider Cauchy problem for kinematic equation concerning motion of a particle

$$\frac{dy_\varepsilon}{dt} = \vec{u}_\varepsilon(y_\varepsilon,t),\ (y_\varepsilon,t) \in Q_T, \quad y_\varepsilon|_{t=t_1} = x.$$

The mapping $\Phi_\varepsilon^{t_1,t_2} : \Omega \to \Omega$ is defined by the formula $\Phi_\varepsilon^{t_1,t_2}(x) = y_\varepsilon(t_2)$.

Since $\vec{u}_\varepsilon \in L_\infty(0,T;C(\Omega))$ in view of Lemma 3.2, $\operatorname{div} \vec{u}_\varepsilon = 0$, and $\vec{u}_\varepsilon|_{\partial\Omega} = 0$, due to Carathéodory theorem [10, Chapter II, §5.3] this mapping is absolutely continuous in Q_T for every $\varepsilon > 0$, and, in view of Euler's formula [11, Chapter II, §5, Formula (II.5-8)], Jacobian J of the transformation $x \to \Phi_\varepsilon^{t_1,t_2}(x)$ is identically equal to one for all $t_1, t_2 \in [0,T]$. Hence, the solution $\mu_\varepsilon(x,t)$ of the transport equation (1.8) admits the representation in terms of operator $\Phi_\varepsilon^{t_1,t_2}$ in the form

$$\mu_\varepsilon(x,t) = \mu_\varepsilon^0(\Phi_\varepsilon^{t,0}(x)) = \begin{cases} 1, & x \in \Phi_\varepsilon^{0,t}(\Omega \setminus V_0), \\ 1/\varepsilon, & x \in \Phi_\varepsilon^{0,t}(V_0), \end{cases} \tag{3.5}$$

Denote $V_\varepsilon(t) = \Phi_\varepsilon^{0,t}(V_0)$. Consider the Cauchy problem for transport equation

$$D_t\Lambda_\varepsilon + \vec{u}_\varepsilon \nabla \Lambda_\varepsilon = 0, \quad \Lambda_\varepsilon(x,0) = \Lambda^0(x) = \begin{cases} 1, & x \in V_0, \\ 0, & x \in \Omega \setminus V_0. \end{cases} \tag{3.6}$$

Its solution is understood in sense of the integral identity

$$\int_{Q_T} \Lambda_\varepsilon(D_t\psi + \vec{u}_\varepsilon \nabla\psi)dxdt + \int_\Omega \Lambda^0 \psi(x,0)dx = 0, \tag{3.7}$$

where ψ is a test function satisfying $\psi \in C^1(Q_T)$, $\psi|_{t=T} = 0$.

Solution of the equation (3.6) admits the representation

$$\Lambda_\varepsilon(x,t) = \Lambda_\varepsilon^0(\Phi_\varepsilon^{t,0}(x)) = \begin{cases} 1, & x \in V_\varepsilon(t), \\ 0, & x \in \Omega \setminus V_\varepsilon(t). \end{cases}$$

Futher on, we will use the transport equation (3.6) in order to describe the evolution of the set V_ε because it allows to formulate conveniently the question about motion of bodies in terms of functions belonging to the class $\mathrm{Char}(\Omega)$.

The following property of the shift operator is of great importance for investigation of limiting transition as $\varepsilon \to 0$.

Lemma 3.3. *(Hölder's continuity of the shift operator). Let $\vec{u}(x,t)$ be satisfying the conditions*

$$\|\vec{u}\|_{L_\infty(0,T;C(\Omega))} \le C_{16}, \quad |\vec{u}(x',t) - \vec{u}(x'',t)| \le C_{17}M|x' - x''|(\ln|x' - x''| + 1) \text{ for a. e. } t \in [0,T].$$

Then for rather close to each other moments of time $t, \tau \in [0,T]$ and rather close to each other points of space $x_1, x_2 \in \Omega$ the following bounds are valid

$$|\Phi^{t,0}(x_1) - \Phi^{t,0}(x_2)| \le K|x_1 - x_2|^\delta, \quad |\Phi^{t,0}(x) - \Phi^{\tau,0}(x)| \le K|t-\tau|^\delta,\ x \in \Omega,$$

$$|\Phi^{0,t}(x_1) - \Phi^{0,t}(x_2)| \le K|x_1 - x_2|^\delta, \quad |\Phi^{0,t}(x) - \Phi^{0,\tau}(x)| \le K|t-\tau|^\delta,\ x \in \Omega,$$

where $K = \exp(1 - e^{-C_{16}T})$, $\delta = \exp(-C_{17}MT)$.

Proof of the lemma in the case $\Omega \subset \mathbb{R}^2$ can be found in [5]. For our case $\Omega \subset \mathbb{R}^3$, it is sufficiently to repeat the arguments from [5]. □

3.2.3 Passage to limit as $\varepsilon \to 0$

In strength of Lemma 3.2 we have

$$\|\vec{u}_\varepsilon\|_{L_\infty(0,T;V)} \le C_{16}(p). \tag{3.8}$$

Hence, extracting the proper subsequence we arrive at

$$\vec{u}_\varepsilon \to \vec{u} \text{ weak-star in } L_\infty(0,T;V). \tag{3.9}$$

Due to Lemma 3.3 the sets $\{\Phi_\varepsilon^{t,0}(x)\}_{\varepsilon>0}$, $\{\Phi_\varepsilon^{0,t}(x)\}_{\varepsilon>0}$ consisting of $\Phi_\varepsilon^{t,0}(x)$, $\Phi_\varepsilon^{0,t}(x) \in C(Q_T;\, \mathbb{R}^3)$, consequently, are uniformly bounded and equicontinuous with respect to ε. Hence, we conclude

$$\Phi_\varepsilon^{t,0}(x) \to \Phi^{t,0}(x) \text{ in } C(Q_T), \tag{3.10}$$

$$\Phi_\varepsilon^{0,t}(x) \to \Phi^{0,t}(x) \text{ in } C(Q_T), \tag{3.11}$$

where operator Φ^{t_1,t_2} is associated with vector field $\vec{u}(x,t)$. These limiting expressions yield

$$\Lambda_\varepsilon \to \Lambda \text{ in } L_\vartheta(Q_T),\ \vartheta < \infty, \quad \Lambda(x,t) = \Lambda^0(\Phi^{t,0}(x)), \tag{3.12}$$

since $\Lambda_\varepsilon(x,t) = \Lambda^0(\Phi_\varepsilon^{t,0}(x)) \in L_\vartheta(Q_T)$ and the formula [12, Chapter 3, §2.2]

$$\|\Lambda^0(\Phi^{t,0}(x) + (\Phi_\varepsilon^{t,0}(x) - \Phi^{t,0}(x))) - \Lambda^0(\Phi^{t,0}(x))\|_{\vartheta,\Omega} \to 0$$

is valid. Further on, we will utilize the following statement.

Lemma 3.4. *For any $\sigma > 0$ there exists $\varepsilon_0 > 0$ such that $S(\Lambda_\varepsilon(t)) \subset S_\sigma(\Lambda(t))$ for all $\varepsilon < \varepsilon_0$ and $t \in [0,T]$.*

In strength of (3.11), assertion of the lemma is valid due to the representation $S(\Lambda_\varepsilon(t)) = \Phi_\varepsilon^{0,t}(S(\Lambda^0))$. □

In order to fulfil limiting transition as $\varepsilon \to 0$ in the inequality (1.7), it is necessary to obtain additional compactness property for $\{\vec{u}_\varepsilon\}$.

Let X be a Hilbert space supplied with inner product $(\cdot,\cdot)_X$, Y be a closed subspace of X, what particularly means that Y is a Hilbert space supplied with inner product $(\cdot,\cdot)_X$. Let P is an orthogonal projector from X in Y, assume also $PX = Y$. Let X_0, X_1 be Banach spaces such that $X_0 \subset X$, $Y \subset X_1$. Let these embeddings be dense. Besides, let $X_0 \subset X$ compactly. In view of all these suppositions, the following lemma is true [2].

Lemma 3.5. *Let $\{\vec{v}_k\}$ be a sequence satisfying*

$$\|\vec{v}_k\|_{L_p(0,T;X_0)} \le C, \quad \|D_t(P\vec{v}_k)\|_{L_{p'}(0,T;X_1)} \le C,$$

$1 < p < \infty$. *Then the sequence $\{P\vec{v}_k\}$ is compact in $L_p(0,T;X)$.*

Introduce $G^\sigma = \{(x,t) \in Q_T \,|\, x \in \Omega \setminus S_\sigma(\Lambda(t)),\ t \in [0,T]\}$. Fix $\sigma > 0$ and consider an arbitrary cilinder $E = A \times [t_1,t_2]$, $t_1,t_2 \in [0,T]$, $E \subset G^\sigma$. Let P_A: $H(A) \to H_0(A)$ be an orthogonal projector. Any function $\vec{v} \in V(A)$ admits the representation [13]

$$\vec{v} = P_A\vec{v} + \nabla\alpha_A, \text{ where } \Delta\alpha_A = 0 \ (x \in A). \tag{3.13}$$

Proposition 3.1. *Sequence $\{D_t(P_A\vec{u}_\varepsilon)\}$ is uniformly bounded in $L_{p'}(t_1,t_2,V^{-3}(A))$.*

Proof. Turn back to investigation of Problem A. Substituting into (2.2) vector $\vec{v} \in V^3(A) \cap H_0(A)$ on the place of test function we arrive at

$$\int_A (D_t P_A \vec{u}_\varepsilon)\vec{v}dx + \int_A \vec{a}(\mu_\varepsilon, \vec{u}_\varepsilon)\vec{v}dx + \int_A \vec{u}_\varepsilon \otimes \vec{v} : \nabla\vec{u}_\varepsilon dx + \frac{1}{\varepsilon}\int_A \beta(\vec{u}_\varepsilon)\vec{v}dx = \int_A \vec{f}\vec{v}dx, \tag{3.14}$$

because

$$\int_A (D_t\vec{u})\vec{v}dx = \int_A (D_t P_A\vec{u}_\varepsilon + D_t\nabla\alpha_A^\varepsilon)\vec{v}dx = \int_A (D_t P_A\vec{u}_\varepsilon)\vec{v}dx - \int_A (D_t\alpha_A^\varepsilon)\vec{v}dx, \quad \mathrm{div}\vec{v} = 0.$$

Note that here ε is a parameter from outlinings that concern Problem A.

Integrate (3.14) with respect to t over $(t_1,\, t_2)$. Following the arguments in Subsection 2.2 we establish

$$\|D_t P_A \vec{u}_\varepsilon\|_{L_{p'}(t_1,t_2,V^{-3}(A))} \le C_8 + C_s^{(1)}(\Omega) C_4^{p'} \|\mu_\varepsilon\|^{1/p'}_{C([t_1,t_2];L_p(A))} + C_{10} + \|\vec{f}\|_{L_{p'}(0,T;V')} C_s^{(2)}(\Omega,p).$$

Here, passing to limit as $\varepsilon \to 0$ and using the formula (2.40) we conclude that the following estimate for solution of Problem A takes place.

$$\|D_t P_A \vec{u}\|_{L_{p'}(t_1,t_2,V^{-3}(A))} \le C_8 + C_s^{(1)}(\Omega) C_4^{p'} \|\mu\|^{1/p'}_{C([t_1,t_2];L_p(A))} + C_{10} + \|\vec{f}\|_{L_{p'}(0,T;V')} C_s^{(2)}(\Omega,p). \tag{3.15}$$

Turn back to studying of Problem B. Observe that the constants C_4, C_8 and C_{10} do not depend on $\operatorname{ess\,sup}\mu$. Due to Lemma 3.4 and the choice of the set E we have $\mu_\varepsilon = 1$ on E for all $\varepsilon < \varepsilon_0$. Hence, $\|\mu_\varepsilon\|_{C([t_1,t_2];L_p(A))} \le (\operatorname{meas} A)^{1/p}$. Due to this the assertion of the proposition flows from the formula (3.15).

In strength of (3.9) and (3.12), limiting transition as $\varepsilon \to 0$ in the integral identity (3.7) does not cause any difficulties, and we get

$$\int_{Q_T} \Lambda(D_t\psi + \vec{u}\nabla\psi)dxdt + \int_\Omega \Lambda^0 \psi(x,0)dx = 0, \tag{3.16}$$

where $\psi \in C^1(Q_T)$ is a test function satisfying $\psi|_{t=T} = 0$. Besides, we have

$$\Lambda \in \operatorname{Char}(Q_T). \tag{3.17}$$

Now, we should pass to limit in the inequality

$$\int_{Q_T} D_t\vec{\varphi}(\vec{\varphi} - \vec{u}_\varepsilon)dxdt + \int_{Q_T} \mu_\varepsilon |\mathbb{D}(\vec{u}_\varepsilon)|^{p-2}\mathbb{D}(\vec{u}_\varepsilon) : \mathbb{D}(\vec{\varphi} - \vec{u}_\varepsilon)dxdt$$
$$- \int_{Q_T} \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon : \nabla(\vec{\varphi} - \vec{u}_\varepsilon)dxdt \ge \int_{Q_T} \vec{f}(\vec{\varphi} - \vec{u}_\varepsilon)dxdt. \tag{3.18}$$

Recall that test function $\vec{\varphi}$ satisfies the conditions defined in (2.44), that is either $\vec{\varphi}(0) = \vec{u}_0$ or $\|\vec{\varphi}(T)\|^2_{2,\Omega} \ge C_6^2 - \|\vec{\varphi}(0) \pm \vec{u}_0\|^2_{2,\Omega}$.

In order to pass to the limit in nonlinear term due to Lemma 3.4 it is enough to show that

$$\lim_{\varepsilon\to 0} \int_{Q_T} (\vec{u}_\varepsilon \otimes \vec{u}_\varepsilon) : \mathbb{D}(\vec{\varphi})dxdt = \int_{Q_T} (\vec{u} \otimes \vec{u}) : \mathbb{D}(\vec{\varphi})dxdt \tag{3.19}$$

for any smooth test function $\vec{\varphi}$ such that $\mathbb{D}(\vec{\varphi}) = 0$ if $x \in S_\sigma(\Lambda)$, $\sigma > 0$. Let us repeat the arguments from [2] that have been done with the aim to study the problem similar to ours.

Choose $\vec{\gamma} : Q_T \to \mathbb{R}^3$ such that $\mathbb{D}(\vec{\gamma}) \equiv 0$ in Q_T and $\vec{\gamma}(x,t) = \vec{\varphi}(x,t)$ if $x \in S_\sigma(\Lambda)$. Since the set consisting of $\vec{u}_\varepsilon$ is uniformly bounded in $L_2(0,T;V^1(\Omega))$ we have due to [7, Chapter 1, Lemma 7.1] that for all $\delta > 0$, $\beta > 0$ there exists the function $\vec{g} \in L_2(0,T;V^1(\omega))$ with the properties $\vec{g}(x,t) = 0$ if $x \in \Omega \setminus (\partial\Omega)_\delta$, $\vec{g}(x,t) = \vec{\gamma}(x,t)$ if $x \in \partial\Omega$, and

$$\left| \int_{Q_T} (\vec{u}_\varepsilon \otimes \vec{u}_\varepsilon) : \mathbb{D}(\vec{g})dxdt \right| \le \beta \|\vec{u}_\varepsilon\|^2_{L_2(0,T;V^1(\Omega))}. \tag{3.20}$$

Let $\vec{w} = \vec{\varphi} - \vec{\gamma}$. We have

$$\int_{Q_T} (\vec{u}_\varepsilon \otimes \vec{u}_\varepsilon) : \mathbb{D}(\vec{\varphi})dxdt$$
$$= \int_{Q_T} (\vec{u}_\varepsilon \otimes \vec{u}_\varepsilon) : \mathbb{D}(\vec{w})dxdt = \int_{Q_T} (\vec{u}_\varepsilon \otimes \vec{u}_\varepsilon) : \mathbb{D}(\vec{w} + \vec{g})dxdt - \int_{Q_T} (\vec{u}_\varepsilon \otimes \vec{u}_\varepsilon) : \mathbb{D}(\vec{g})dxdt. \tag{3.21}$$

Due to the properties of functions $\vec{\varphi}$, $\vec{w}$, $\vec{\gamma}$ and $\vec{g}$ on $S_\sigma(\Lambda)$ and on $\partial\Omega$, the equality $\vec{w} + \vec{g} = 0$ takes place (provided with the proper choice of δ). Also, last term in (3.21) can be done arbitrary small in strength of arbitrariness of β. These conclusions mean that it is sufficiently to verify the equality (3.19) just for the test functions $\vec{\varphi}$ that vanish on $S_\sigma(\Lambda)$ and on $\partial\Omega$.

In strength of Proposition 3.1 and Lemma 2.5, $P_A\vec{u}_\varepsilon \to P_A\vec{u}$ in $L_p([t_1,t_2], H(A))$. Besides, $\nabla\alpha_A^\varepsilon \to \nabla\alpha_A$ weakly in $L_p([t_1,t_2], L_2(A))$ and $\vec{u} = P_A\vec{u} + \nabla\alpha_A$ if $(x,t) \in E$. Hence,

$$\lim_{\varepsilon\to 0} \int_E (P_A\vec{u}_\varepsilon \otimes P_A\vec{u}_\varepsilon) : \mathbb{D}(\vec{\varphi})dxdt = \int_E (P_A\vec{u} \otimes P_A\vec{u}) : \mathbb{D}(\vec{\varphi})dxdt$$

for any $\vec{\varphi} \in L_p(0,T;V)$, $\vec{\varphi} = 0$ if $(t,x) \notin E$.

Let us show that

$$\int_E (\nabla\alpha_A^\varepsilon \otimes \nabla\alpha_A^\varepsilon) : \mathbb{D}(\vec{\varphi})dxdt = \int_E (\nabla\alpha_A \otimes \nabla\alpha_A) : \mathbb{D}(\vec{\varphi})dxdt = 0.$$

Indeed, as far as $\vec{\varphi}(x,t)=0$ if $x\in\partial A$, $\mathrm{div}\vec{\varphi}=0$, we have

$$\int_A(\nabla\alpha_A^\varepsilon\otimes\nabla\alpha_A^\varepsilon):\mathbb{D}(\vec{\varphi})dx=-\int_A\mathrm{div}\,(\nabla\alpha_A^\varepsilon\otimes\nabla\alpha_A^\varepsilon)\cdot\vec{\varphi})dx$$
$$=-\int_A(\Delta\alpha_A\nabla\alpha_A+\frac{1}{2}\nabla|\nabla\alpha_A|^2)\cdot\vec{\varphi}dx=\int_A\frac{1}{2}|\nabla\alpha_A|^2\mathrm{div}\,\vec{\varphi}dx=0.$$

This yields

$$\lim_{\varepsilon\to0}\int_E(\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon):\mathbb{D}(\vec{\varphi})dxdt=\int_E(\vec{u}\otimes\vec{u}:\mathbb{D}(\vec{\varphi})dxdt.$$

Since E is arbitrary and the set G^σ admits an approximation by a countable set of cilinders like E, we conclude that the formula (3.19) holds true.

In strength of the estimate (3.8) and the formula (3.9) we see

$$\textstyle\int_{Q_T}\vec{\varphi}'(\vec{\varphi}-\vec{u}_\varepsilon)dxdt\to\int_{Q_T}\vec{\varphi}'(\vec{\varphi}-\vec{u})dxdt, \tag{3.22}$$
$$\textstyle\int_{Q_T}\vec{f}(\vec{\varphi}-\vec{u}_\varepsilon)dxdt\to\int_{Q_T}\vec{f}(\vec{\varphi}-\vec{u})dxdt, \tag{3.23}$$
$$\textstyle\int_{Q_T}\vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{\varphi}dxdt\to\int_{Q_T}\chi_{**}\vec{\varphi}dxdt, \tag{3.24}$$

where $\vec{\varphi}\in W_p^1(0,T;V)\cap K_\sigma(\Lambda)$. Substituting into (3.18) some admissible function $\vec{\varphi}\in K_\sigma(\Lambda)$ we arrive at the bound

$$\int_{Q_T}\Lambda_\varepsilon|\mathbb{D}(\vec{u}_\varepsilon)|^pdxdt\le\varepsilon C_{18},\ \text{where } C_{18}=C_{18}(C_{16}(p),\,\|\vec{\varphi}\|_{W_p^1(0,T;V)}), \tag{3.25}$$

or, equivalently,

$$\int_0^T\int_{S(\Lambda_\varepsilon)}|\mathbb{D}(\vec{u}_\varepsilon)|^pdxdt\le\varepsilon C_{18}. \tag{3.26}$$

Due to Lemma 3.4 we deduce from (3.26) that

$$\vec{u}\in K(\Lambda)\text{ for a. e. } t\in[0,T]. \tag{3.27}$$

Consider

$$\int_{Q_T}\vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{u}_\varepsilon dxdt=\int_{Q_T}\mu_\varepsilon|\mathbb{D}(\vec{u}_\varepsilon)|^pdxdt=\int_{Q_T}(1-\Lambda_\varepsilon)|\mathbb{D}(\vec{u}_\varepsilon)|^pdxdt+\int_{Q_T}\Lambda_\varepsilon\varepsilon^{-1}|\mathbb{D}(\vec{u}_\varepsilon)|^pdxdt$$
$$\ge\int_{Q_T}(1-\Lambda_\varepsilon)|\mathbb{D}(\vec{u}_\varepsilon)|^pdxdt\ge\int_{Q_T}|\mathbb{D}(\vec{u}_\varepsilon)|^pdxdt-\varepsilon C_{19}.$$

Passing to limit as $\varepsilon\to0$ in strength of (3.8) and Proposition 2.2, we see

$$\liminf_{\varepsilon}\int_{Q_T}\vec{a}(\mu_\varepsilon,\vec{u}_\varepsilon)\vec{u}_\varepsilon dxdt\ge\int_{Q_T}|\mathbb{D}(\vec{u})|^pdxdt. \tag{3.28}$$

Thus, in view of (3.19), (3.22)–(3.24) and (3.28), passing to limit as $\varepsilon\to0$ in (3.18) we establish

$$\int_{Q_T}\vec{\varphi}'(\vec{\varphi}-\vec{u})dxdt+\int_{Q_T}\chi_{**}\vec{\varphi}dxdt-\int_{Q_T}|\mathbb{D}(\vec{u})|^pdxdt-\int_{Q_T}\vec{u}\otimes\vec{u}:\nabla(\vec{\varphi}-\vec{u})dxdt\ge\int_{Q_T}\vec{f}(\vec{\varphi}-\vec{u})dxdt \tag{3.29}$$

for all $\vec{\varphi}\in W_p^1(0,T;V)\cap K_\sigma(\Lambda)$ satisfying the bound $|\mathbb{D}(\vec{\varphi})|\le M$ and the condition (2.44). Since $\sigma>0$ is arbitrary, and surfaces $\partial S(\Lambda)$ of solid bodies are Lipschitz continuous we conclude on the base of [2, Proposition 2.1] that the inequality (3.29) hold true for $\vec{\varphi}\in K(\Lambda)$ as well.

Finally, repeating the arguments from Subsection 2.4 we get $\chi_{**}=\vec{a}(1,\vec{u})$.

Hence, the integral inequality (1.19) holds true. Proof of Theorem 3.1 is completed.

4 Appendix

4.1 Consistency of Definitions 1.1 and 1.2

Here, we outline explanations concerning the definitions of generalized solutions to Problems A and B. Namely, the following statements will be proved.

Proposition 4.1. *Let generalized solution* $\{\vec{u}, \mu\}$ *of Problem A be smooth. Then* (1.7) *and* (1.8) *yield the identities* (1.1) *and* (1.2).

Proposition 4.2. *Let in generalized solution* $\{\vec{u}, \Lambda\}$ *of Problem B vector-function* $\vec{u}$ *be smooth. Then the inequality* (1.19) *yields the identities* (1.9), (1.11) *and* (1.12).

PROOF OF PROPOSITION 4.1. Easy to see that, if $\vec{v}, \vec{u}|_{\partial\Omega} = 0$, then the equality

$$\int_\Omega \mu W : \mathbb{D}(\vec{v} - \vec{u}) dx = -\int_\Omega \operatorname{div}(\mu W)(\vec{v} - \vec{u}) dx,$$

holds true. Thus, if $\mu W \in \partial\Phi(\mathbb{D}(\vec{u}))$ then $-\operatorname{div}(\mu W) \in \partial\Phi^{(1)}(\vec{u})$, where the functional $\Phi^{(1)}$ has a form $\Phi^{(1)}(\vec{v}) = \frac{1}{p}\int_\Omega Q(x, \vec{v}(x)) dx$. Here,

$$Q(x, \vec{v}(x)) = \begin{cases} \mu(x)|\mathbb{D}(\vec{v}(x))|^p, & \text{if } |\mathbb{D}(\vec{v})| \le M, \\ +\infty, & \text{if } |\mathbb{D}(\vec{v})| > M. \end{cases}$$

So, the equality (1.1) is equivalent to the following one.

$$-(D_t\vec{u} + \sum_{i=1}^{3} u_i D_i \vec{u} + \nabla p_* - \vec{f}) \in \partial\Phi^{(1)}(\vec{u}). \tag{4.1}$$

Now, consider the inequality (1.7) which can be written in the form

$$\int_{Q_T} \vec{u}'(\vec{\varphi} - \vec{u}) dxdt + \int_{Q_T} (\vec{\varphi}' - \vec{u}')(\vec{\varphi} - \vec{u}) dxdt - \int_{Q_T} \operatorname{div}(\mu|\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}))(\vec{\varphi} - \vec{u}) dxdt$$
$$- \int_{Q_T} \vec{u} \otimes \vec{u} : \nabla(\vec{\varphi} - \vec{u}) dxdt \ge \int_{Q_T} \vec{f}(\vec{\varphi} - \vec{u}) dxdt.$$

The latter inequality yields that, if $\|\vec{\varphi}(T) - \vec{u}(T)\|_{2,\Omega} = 0$, then

$$-\int_{Q_T} \operatorname{div}(\mu|\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}))(\vec{\varphi} - \vec{u}) dxdt \ge -\int_{Q_T} (D_t\vec{u} + \sum_{i=1}^{3} u_i D_i \vec{u} + \nabla p_* - \vec{f})(\vec{\varphi} - \vec{u}) dxdt. \tag{4.2}$$

Note that,if $|\mathbb{D}(\vec{u})| \le M$, then $\{\Phi^{(1)\prime}(\vec{u})\} = \partial\Phi^{(1)}(\vec{u})$ where $\Phi^{(1)\prime} = \operatorname{div}(\mu|\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}))$ is the Gâteaux derivative of $\Phi^{(1)}$ at a point $\vec{u}$ [14]. Hence, in strength of (4.2), the inclusion (4.1) is valid, what amounts to the fact that the equality (1.1) hold true. Finally, the equality (1.2) is a simple consequence of the integral identity (1.8). □

PROOF OF PROPOSITION 4.2. Since $\vec{\varphi}$, $\vec{u} = 0$ if $x \in \partial\Omega$, and $\vec{u}$, $\vec{\varphi} \in K(\Lambda)$, the equality

$$\int_0^T dt \int_\Omega |\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}) : \mathbb{D}(\vec{\varphi} - \vec{u}) dx$$
$$= -\int_0^T dt \int_{\partial V(t)} (|\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u})\vec{n}) : (\vec{\varphi} - \vec{u}) d\sigma - \int_0^T dt \int_{\Omega\setminus V(t)} \operatorname{div}(|\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}))(\vec{\varphi} - \vec{u}) dx \tag{4.3}$$

holds true. Here, $\vec{n}$ is the unit normal to $\partial V(t)$ directed into $\Omega \setminus V(t)$. Proceeding the arguments from the proof of Proposition 4.1 and using the formula (4.3) we deduce from (1.19) the following.

$$-\int_0^T dt \int_{\partial V(t)} ([-Ip_* + |\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u})]\vec{n})(\vec{\varphi} - \vec{u}) d\sigma - \int_0^T dt \int_{\Omega\setminus V(t)} \operatorname{div}(-Ip_* + |\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}))(\vec{\varphi} - \vec{u}) dx$$
$$\ge -\int_{Q_T} (D_t\vec{u} + \sum_{i=1}^{3} u_i D_i \vec{u} - \vec{f})(\vec{\varphi} - \vec{u}) dxdt. \tag{4.4}$$

Introducing into (4.4) on the place of a test function a function $\vec{\varphi}$ which we demand to satisfy $\vec{\varphi} = \vec{u}$ on V_T we arrive at

$$-\int_0^T dt \int_{\Omega\setminus V(t)} \operatorname{div}(-Ip_* + |\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}))(\vec{\varphi} - \vec{u}) dx \ge -\int_0^T dt \int_{\Omega\setminus V(t)} (D_t\vec{u} + \sum_{i=1}^{3} u_i D_i \vec{u} - \vec{f})(\vec{\varphi} - \vec{u}) dx. \tag{4.5}$$

Repeating the arguments from the proof of Proposition 4.1 we deduce the equality (1.9) from (4.5).

Next, since $\vec{u}$ is in $K(\Lambda)$ and is smooth there exists $\delta > 0$ such that $|\mathbb{D}(\vec{u})| \leq M/2$ if $x \in S_\delta(\Lambda)$.

Impose in (4.4) and (4.5) that $\vec{\varphi} = \vec{u} \pm \vec{\psi}$ where supp$\vec{\psi} \in S_\delta(\Lambda)$, $|\mathbb{D}(\vec{\psi})| \leq M/2$, $\vec{\psi} \in K(\Lambda)$, $\vec{\psi}|_{t=0} = 0$. Clearly, such choise of test function is legal. Due to arbitrariness of $\vec{\psi}$ (within the imposed demands) the formulae (4.4) and (4.5) yield

$$-\int_0^T dt \int_{\partial V(t)} ([-Ip_* + |\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u})]\vec{n})\vec{\psi} d\sigma = \int_0^T dt (D_t\vec{u} + \sum_{i=1}^3 u_i D_i \vec{u} - \vec{f})\vec{\psi} dx. \tag{4.6}$$

We have $|\mathbb{D}(\vec{u})|^{p-2}\mathbb{D}(\vec{u}) \in \partial\Phi(\mathbb{D}(\vec{u}))$ because $|\mathbb{D}(\vec{u})| \leq M$. Hence, (4.6) gives the following.

$$\int_0^T dt \int_{V(t)} (D_t\vec{u} + \sum_{i=1}^3 u_i D_i\vec{u})\vec{\psi} dx = \int_0^T dt \int_{\partial V(t)} (\mathbb{T}\vec{n})\vec{\psi} d\sigma + \int_0^T dt \int_{V(t)} \vec{f}\vec{\psi} dx, \tag{4.7}$$

where $\mathbb{T}$ is the stress tensor in the fluid, $\mathbb{T} = -Ip_* + W$.

Substituting into (4.7) the explicit forms

$$\vec{u} = \vec{v}_c^{(l)}(t) + \vec{\omega}^{(l)}(t) \times x, \quad \vec{\psi} = \vec{\psi}_c^{(l)}(t) + \vec{\psi}_\omega^{(l)}(t) \times x, \quad (x,t) \in V^{(l)}(t),$$

$l = 1, \ldots, N$ (N is the quantity of floating solid bodies) of $\vec{u}$, $\vec{\psi} \in K(\Lambda)$ (see Remark 1.2), due to arbitrariness of $\vec{\psi}_c^{(l)}(t)$ and $\vec{\psi}_\omega^{(l)}(t)$ we get (1.11) and (1.12) from (4.7). □

4.2 On properties of interactions between solid bodies and between a solid body and $\partial\Omega$

Due to Lemma 3.1, velocity field $\vec{u}(x,t)$ is almost-Lipschitz continuous, i. e.

$$|\vec{u}(x',t) - \vec{u}(x'',t)| \leq C_{12}M|x' - x''|(\ln|x' - x''| + 1) \equiv \tilde{\varphi}(|x' - x''|)$$

for a. e. $t \in [0,T]$, for rather close to each other points x', x''. Suppose $|x' - x''| \leq a =$ const. Since $\int_\varepsilon^a \frac{dv}{\tilde{\varphi}(v)} \to \infty$ as $\varepsilon \to 0$, due to Osgood uniqueness theorem every point $(x_0,\, t_0)$ in Q_T holds not more than just one integral curve of the equation

$$\frac{dy}{dt} = \vec{u}(y,t), \quad (y,t) \in Q_T. \tag{4.8}$$

Hence, if $x_0 \in V^{(1)}(t_0)$, $x_0 \in V^{(2)}(t_0)$, $t_0 \in [0,T]$, then $y(t) \in V^{(1)}(t) \cap V^{(2)}(t)$ for all $t \in [0,T]$ where $y(t)$ is the solution of the equation (4.8) provided with Cauchy data $y(t_0) = x_0$. Thus, *if two bodies get in touch with each other at some moment of time* $t_0 \in [0,T]$ *then they are in touch during the whole interval* $[0,T]$.

In strength of $\vec{u} \in K(\Lambda)$ and due to Remark 1.2, the equalities $\vec{u}(x,t) = \vec{u}_c^{(1)}(t) + \vec{\omega}^{(1)}(t) \times (x - x_c^{(1)}(t))$, $(x,t) \in V_T^{(1)}$, $\vec{u}(x,t) = \vec{u}_c^{(2)}(t) + \vec{\omega}^{(2)}(t) \times (x - x_c^{(2)}(t))$, $(x,t) \in V_T^{(2)}$ hold true. Observe that gradient of velocity

$$\nabla_x \vec{u} = \begin{pmatrix} 0 & \omega_3^{(i)} & -\omega_2^{(i)} \\ -\omega_3^{(i)} & 0 & \omega_1^{(i)} \\ \omega_2^{(i)} & -\omega_1^{(i)} & 0 \end{pmatrix}, \quad (x,t) \in V_T^{(i)}, \quad i = 1,2$$

does not depend x in the case $x \in V$. Since $V^{(1)}(t)$ and $V^{(2)}(t)$ have a common point $(y(t),t)$ these representations yield that

$$\vec{\omega}^{(1)}(t) = \vec{\omega}^{(2)}(t), \quad t \in [0,T].$$

Thus, *two bodies that are in touch with each other have zero relative speed, i. e. they move like an entire solid.* As well, from the above investigations we see that, *if at some moment of time body does not touch other bodies and the boundary* $\partial\Omega$*, then it touches nothing during the whole interval of time* $[0,T]$*. If body touches* $\partial\Omega$ *at some moment then it is immovable during the whole interval* $[0,T]$.

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