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THE GENUINELY NONLINEAR GRAETZ–NUSSELT ULTRAPARABOLIC EQUATION

S. A. Sazhenkov UDC 517.9

Abstract: We study a second-order quasilinear ultraparabolic equation whose matrix of the coefficients of the second derivatives is nonnegative, depends on the time and spatial variables, and can change rank in the case when it is diagonal and the coefficients of the first derivatives can be discontinuous. We prove that if the equation is a priori known to enjoy the maximum principle and satisfies the additional "genuine nonlinearity" condition then the Cauchy problem with arbitrary bounded initial data has at least one entropy solution and every uniformly bounded set of entropy solutions is relatively compact in $L^1_{\rm loc}$. The proofs are based on introduction and systematic study of the kinetic formulation of the equation in question and application of the modification of the Tartar H-measures proposed by E. Yu. Panov.

Keywords: genuine nonlinearity, ultraparabolic equation, entropy solution, anisotropic diffusion

§ 1. Introduction

We consider the Cauchy problem for the quasilinear diffusion-convection equation of the form

$$u_t + \partial_{x_i} a_i(\mathbf{x},t,u) - \partial_{x_i}(a_{ij}(\mathbf{x},t)\partial_{x_j} b(u)) = 0, \quad \mathbf{x}\in\mathbb{R}^d,\ t\in(0,T), \tag{1a}$$

with the initial data in $L^\infty(\mathbb{R}^d)$,

$$u|_{t=0} = u_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \tag{1b}$$

In (1), u is the sought function, while the flow vector $\mathbf{a} := (a_i)$, the diffusion matrix $A := (a_{ij})$, and the diffusion function b are given and satisfy the conditions

$$a_i, D_{x_i}a_i \in L^4_{\rm loc}\big(\mathbb{R}^d_x\times(0,T); C^1_{\rm loc}(\mathbb{R}_u)\big), \quad a_{ij}\in C^2_{\rm loc}\big(\mathbb{R}^d_x\times[0,T]\big), \tag{2}$$

$$a_{ij} = a_{ji}, \quad a_{ij}(\mathbf{x},t)\xi_i\xi_j \ge 0 \quad \forall\,\xi,\ \mathbf{x}\in\mathbb{R}^d,\ t\in[0,T], \tag{3}$$

$$b\in C^2_{\rm loc}(\mathbb{R}), \quad b'(u) > 0 \quad \forall\,u\in\mathbb{R}. \tag{4}$$

We assume a priori that the maximum principle holds for (1); i.e., for example, the inequality

$$uD_{x_i}a_i(\mathbf{x},t,u) \ge -c_1u^2 - c_2 \quad \text{for a.e. } \mathbf{x}\in\mathbb{R}^d,\ t\in[0,T]\ \forall\,u\in\mathbb{R} \tag{5}$$

holds for given functions with some positive constants c_1 and c_2 [1, Chapter I, Theorem 2.9].

In (1)–(5) and below we use the conventional rule of summation over repeated indices in products. The derivative D_{x_i} is defined by the formula

$$D_{x_i}g(\mathbf{x},t,u) = (\partial_{x_i}g(\mathbf{x},t,\lambda))|_{\lambda=u(x,t)} \quad \forall\,g\in C^1\big(\mathbb{R}^d_x\times(0,T)\times\mathbb{R}_\lambda\big).$$

In particular, the derivatives ∂_{x_i} and D_{x_i} are connected by the identity

$$\partial_{x_i}g(\mathbf{x},t,u) = D_{x_i}g(\mathbf{x},t,u) + \partial_u g(\mathbf{x},t,u)\partial_{x_i}u.$$

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We suppose that in the general case the rank d_0 of A can be less than the dimension of $\mathbb{R}^d_x$ and can vary with $\mathbf{x}$ and t in the case when A is a diagonal matrix, i.e., $A = \operatorname{diag}(a_{11}(\mathbf{x}, t), \dots, a_{dd}(\mathbf{x}, t))$. Thus, (1a) is an ultraparabolic equation. Such equations arise in fluid dynamics, combustion theory, and financial mathematics (see the survey [2]). They describe in particular nonstationary transport of substance or heat in the case when the diffusion effect in some spatial directions is negligibly small as compared with convection [3]. These equations were first considered in Graetz's article [4] and Nusselt's article [5].

In this article, under the additional condition of genuine nonlinearity (see Condition G below), using the kinetic equation method and the notion of Tartar H-measure, we prove existence of a bounded entropy solution to the Cauchy problem for (1a) and the relative compactness in L^1_{loc} of every set of entropy solutions to (1a) uniformly bounded in the norm of L^∞.

To state the notion of a bounded entropy solution, we introduce the notation $\Pi := \mathbb{R}^d_x \times (0, T)$ and note that, since (1a) is degenerate, the gradient $\nabla_x u$ of a possible solution $u \in L^\infty(\Pi)$ with respect to the spatial variable can be understood only in the distributional sense. However, since A is symmetric and nonnegative, A has a unique square root $A^{1/2} = \{\alpha_{ij}\}_{i,j=1,\dots,d}$ which is a symmetric and nonnegative matrix, too. Hence, applying the well-known methods for construction of a priori estimates for parabolic equations (see [1, Chapter III, § 2; Chapter V, § 1]) and carrying out the formal derivation of the first energy inequality for (1a), we can conclude that, for every function $\Phi \in C_0^\infty(\Pi)$, a possible solution u to (1) satisfies a priori the inequality

$$\|A^{1/2}\nabla_x(\Phi u)\|_{L^2(\Pi)} \le c(\Phi),$$

where the constant $c = c(\Phi)$ is independent of u. This means that, although a single derivative $\partial_{x_i} u$ for some or even all $i = 1, \dots, d$ may fail to be locally summable on Π, the linear combinations of these derivatives of the form $\alpha_{ij}\partial_{x_j} u$ belong to $L^2_{\text{loc}}(\Pi)$. Therefore, the definition of an entropy solution should be supplemented with the corresponding requirement of partial summability of $\nabla_x u$.

Now, we can define the notion of a bounded entropy solution to (1).

Definition 1. A function $u = u(\mathbf{x}, t)$ is an *entropy solution* to (1) if it satisfies the conditions

$$u \in L^\infty(\Pi), \quad \alpha_{ij}\partial_{x_j} u \in L^2_{\text{loc}}(\Pi), \quad i = 1, \dots, d, \tag{6}$$

and the integral entropy inequality

$$\begin{aligned}\int\limits_\Pi \big(&\zeta_t \varphi(u) + \zeta_{x_i} q_i(\mathbf{x}, t, u) - \zeta\varphi'(u) D_{x_i} a_i(\mathbf{x}, t, u) \\ &+ \zeta D_{x_i} q_i(\mathbf{x}, t, u) + w(u)\partial_{x_i}(a_{ij}(\mathbf{x}, t)\partial_{x_j}\zeta) \\ &- \zeta\varphi''(u) b'(u)(\alpha_{il}(\mathbf{x}, t)\partial_{x_i} u)(\alpha_{lj}(\mathbf{x}, t)\partial_{x_j} u)\big)\, d\mathbf{x}dt + \int\limits_{\mathbb{R}^d} \varphi(u_0)\zeta(\mathbf{x}, 0)\, d\mathbf{x} \ge 0\end{aligned} \tag{7}$$

for arbitrary functions φ, q_i, and w such that

$$\varphi \in C^2_{\text{loc}}(\mathbb{R}),\ \varphi''(u) \ge 0,\ \partial_u q_i(\mathbf{x}, t, u) = \varphi'(u)\partial_u a_i(\mathbf{x}, t, u),\ w'(u) = \varphi'(u) b'(u), \tag{8}$$

and for every nonnegative function $\zeta \in C^2(\Pi)$ vanishing in a neighborhood of the plane $\{t = T\}$ and at large values of $|\mathbf{x}|$.

Definition 2. A function u is an *entropy solution* to (1a) if it satisfies the integral inequality (7) for every function $\zeta \in C_0^2(\Pi)$.

Alongside (2)–(5), we also impose the following requirement on the functions a_i, a_{ij}, and b called the *genuine nonlinearity* condition:

Condition G. *The following requirement is satisfied for a.e.* $(\mathbf{x}, t) \in \Pi$*: for arbitrary* $(\xi, \tau) \in \mathbb{R}^{d+1}$ *such that* $|\xi|^2 + \tau^2 = 1$ *and* $a_{ij}(\mathbf{x}, t)\xi_i\xi_j = 0$*, the set*

$$\{\lambda \in \mathbb{R} \mid \tau + (a_{i\lambda}(\mathbf{x}, t, \lambda) + (1/2)b'(\lambda)a_{ijx_j}(\mathbf{x}, t))\xi_i = 0\}$$

has Lebesgue measure zero.

Here and in the sequel $\zeta_\lambda = \partial_\lambda \zeta$ and $\zeta_{x_i} = \partial_{x_i}\zeta \ \forall\, \zeta = \zeta(\mathbf{x}, t, \lambda)$. We denote by $\mathbb{S}^d$ the unit sphere in $\mathbb{R}^{d+1}$: $\mathbb{S}^d := \{(\xi, \tau) \in \mathbb{R}^{d+1} \mid |\xi|^2 + \tau^2 = 1\}$.

The following two theorems are the main results of the present article:

Theorem 1. *Let* (1a) *be a genuinely nonlinear equation in the sense of Condition* G*. Suppose that the matrix* A *of the coefficients of the second derivatives satisfies one of the following two conditions:* (1) *the rank* d_0 *of* A *is constant or* (2) A *is diagonal; i.e.,* $A = \operatorname{diag}(a_{11}(\mathbf{x}, t), \dots, a_{dd}(\mathbf{x}, t))$*.*

Then (1) *has a bounded entropy solution for arbitrary initial data* $u_0 \in L^\infty(\mathbb{R}^d)$*.*

Theorem 2. *Let* (1a) *be a genuinely nonlinear equation in the sense of Condition* G*. Suppose that the matrix* A *of the coefficients of the second derivatives satisfies one of the two conditions:* (1) *the rank* d_0 *of* A *is constant or* (2) A *is diagonal; i.e.,* $A = \operatorname{diag}(a_{11}(\mathbf{x}, t), \dots, a_{dd}(\mathbf{x}, t))$*.*

Then every set of bounded entropy solutions to (1a) *bounded in* $L^\infty(\Pi)$ *is relatively compact in* $L^1_{\mathrm{loc}}(\Pi)$*.*

Remark 1. Observe that the condition (2) in the statements of Theorems 1 and 2 does not exclude the case when the rank of $A = \operatorname{diag}(a_{11}(\mathbf{x}, t), \dots, a_{dd}(\mathbf{x}, t))$ is variable.

There are many articles on the genuinely nonlinear equations of a form similar to (1a). One of the first results in this direction was obtained by Lax [6] who proved in 1957 that the Cauchy problem for the equation $u_t + a(u)_x = 0$ has an entropy solution in the case when the function $a = a(u)$ is convex or concave. It was Lax's article where the equations satisfying conditions like Condition G were called *genuinely nonlinear*.

As regards its, this research is close to the articles by Tartar [7], Lions, Perthame, and Tadmor [8], and Panov [9]. In [7] it was demonstrated that every bounded set of entropy solutions to the equation $u_t + \partial_{x_i} a_i(\mathbf{x}, t, u) = 0$, $\mathbf{x} \in \mathbb{R}^2$, is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^2_x \times \mathbb{R}^+)$. In [9] this result was generalized to the case of arbitrary dimension d. In [8] the equations $u_t + \partial_{x_i} a_i(u) = 0$ and $u_t + \partial_{x_i} a_i(u) - \partial^2_{x_i x_j} a_{ij}(u) = 0$ were studied, where the matrix (a'_{ij}) is nonnegative, and similar results on relative compactness in L^1_{loc} were proven. Note that Condition G of this article is a generalization of the genuine nonlinearity conditions in [7–9]. In those articles the genuine nonlinearity conditions were also called the *nondegeneracy conditions*. Note also that for autonomous equations, i.e., for those in which the diffusion matrix $A = A(u)$ and the flow vector $\mathbf{a} = \mathbf{a}(u)$ do not depend explicitly on $\mathbf{x}$ and t, the theory of well-posedness of the Cauchy problems in the classes of bounded entropy solutions was constructed completely and without constraints like the genuine nonlinearity condition in [10, 11]. Equation (1a) of this article is not autonomous in the general case; in this connection, Theorem 1 is a new result on existence of solutions to ultraparabolic equations.

The proofs of Theorems 1 and 2 are based on application of the kinetic equation method which enables us to reduce quasilinear equations to linear scalar equations whose solutions are "distribution" functions containing an additional "kinetic" variable (for example, see [8, 11–13]). Alongside this method, we apply the theory of H-measures originally constructed by Tartar [14] and Gerárd [15] and further developed in Panov's article [9].

§ 2. A Kinetic Formulation of (1a)

We introduce a kinetic formulation for (1a) in a form similar to that of [11].

Problem K. *Find a kinetic function* $f(\mathbf{x}, t, \lambda)$ *and nonnegative Borel measures* $m, n \in \mathbb{M}(\Pi \times \mathbb{R}_\lambda)$ *satisfying the equation*

$$f_t + a_{i\lambda}(\mathbf{x}, t, \lambda) f_{x_i} - a_{ix_i}(\mathbf{x}, t, \lambda) f_\lambda - b'(\lambda)\partial_{x_i}(a_{ij}(\mathbf{x}, t)\partial_{x_j} f) + (m + b'(\lambda)n)_\lambda = 0 \tag{9a}$$

and the conditions

$$f(\mathbf{x},t,\lambda) = \begin{cases} 1 & \text{for } \lambda \geq u(\mathbf{x},t), \\ 0 & \text{for } \lambda < u(\mathbf{x},t), \end{cases} \tag{9b}$$

$$\operatorname{spt} m \subset \{(\mathbf{x},t,\lambda) \in \Pi \times \mathbb{R}_\lambda : |\lambda| \leq \|u\|_{L^\infty}\}, \tag{9c}$$

$$dn(\mathbf{x},t,\lambda) = |A^{1/2}\nabla_x u(\mathbf{x},t)|^2 d\gamma_{u(x,t)}(\lambda)\, d\mathbf{x}dt \tag{9d}$$

with some function $u \in L^\infty(\Pi)$ *such that* $A^{1/2}\nabla_x u \in L^2_{\text{loc}}(\Pi)$.

Here we denote by $\mathbb{M}(X)$ the Banach space of bounded Radon measures on some set X. In (9d) we denote by $\gamma_{u(x,t)}$ the parametrized Dirac measure on $\mathbb{R}_\lambda$ concentrated at $\lambda = u(\mathbf{x},t)$.

The kinetic equation (9a) is understood in the distributional sense, i.e., in the sense of the integral equality

$$\int\limits_{\Pi\times\mathbb{R}_\lambda} (\zeta_t + a_{i\lambda}(\mathbf{x},t,\lambda)\zeta_{x_i} - a_{ix_i}(\mathbf{x},t,\lambda)\partial_\lambda\zeta + b'(\lambda)\partial_{x_i}(a_{ij}(\mathbf{x},t)\partial_{x_j}\zeta))f\, d\mathbf{x}dtd\lambda$$

$$+ \int\limits_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda b'(\lambda)\, dn(\mathbf{x},t,\lambda) + \int\limits_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda\, dm(\mathbf{x},t,\lambda) = 0 \tag{10}$$

in which $\zeta \in C_0^2(\Pi \times \mathbb{R}_\lambda)$ is an arbitrary test function.

Remark 2. By the obvious representation

$$\varphi(u(\mathbf{x},t)) = -\int\limits_{\mathbb{R}} \varphi'(\lambda)f(\mathbf{x},t,\lambda)\, d\lambda \quad \forall\, \varphi \in C_0^1(\mathbb{R}), \tag{11}$$

the triple (f,m,n) is a solution to Problem K if and only if the function u in (9b)–(9d) is a bounded entropy solution to (1a).

The further content of the article is as follows. In § 3 we introduce a family of H-measures corresponding to an arbitrary weakly converging sequence of solutions to Problem K. In § 4 we formulate the localization principle for H-measures (Theorem 3 and Corollary 1) and in § 5–§ 7 prove it. In § 8, using this principle, we prove Theorem 2. In § 9 we prove Theorem 1.

§ 3. Tartar *H*-Measures

Let (f^k, m^k, n^k), $k \in \mathbb{N}$, be a sequence of solutions to Problem K such that the set of values of the variable λ at which f^k has jumps is uniformly bounded in some interval $[-u_*, u_*]$, $u_* = \text{const} > 0$. This means that the corresponding sequence $\{u^k\}$ of entropy solutions to (1a) is uniformly bounded in $L^\infty(\Pi)$ and the estimate $\|u^k\|_{L^\infty(\Pi)} \leq u_*$ is valid. Extracting, if need be, a subsequence of $k \in \mathbb{N}$, we define some weakly* converging subsequences $\{f^k\}$ and $\{u^k\}$ and limit functions $f \in L^\infty(\Pi \times \mathbb{R}_\lambda)$ and $u \in L^\infty(\Pi)$ such that

$$f^k \to f \text{ weakly* in } L^\infty(\Pi \times \mathbb{R}_\lambda) \quad \text{as } k \nearrow \infty, \tag{12}$$

$$u^k \to u \text{ weakly* in } L^\infty(\Pi) \quad \text{as } k \nearrow \infty. \tag{13}$$

It is obvious that $f = 0$ for $\lambda < -u_*$ and $f = 1$ for $\lambda \geq u_*$. The lemma below implies that f is a monotone nondecreasing right continuous function of λ. This structure of f enables us to use Panov's theorem on modification of the notion of Tartar H-measures [9, Theorem 3] and introduce a family of H-measures associated with $f^k - f$.

Lemma 1. *The function f in the limit relation* (12) *is the distribution function of a Young measure $\nu_{x,t} \in \operatorname{Prob}(\mathbb{R}_\lambda)$ associated with the subsequence $\{u^k\}$; i.e.,*

$$f(\mathbf{x},t,\lambda) = \int\limits_{\mathbb{R}_s} 1_{\lambda \geq s}\, d\nu_{x,t}(s). \tag{14}$$

Here $\operatorname{Prob}(\mathbb{R}_\lambda)$ is the subset of $\mathbb{M}(\mathbb{R}_\lambda)$ constituted by all nonnegative measures with the unit norm. The notion of Young measures will be introduced in the proof of the lemma.

Proof. Let $\zeta \in C_0\big(\Pi; C_0^1(\mathbb{R}_\lambda)\big)$ be an arbitrary function. It follows from (12) that

$$\int\limits_{\Pi\times\mathbb{R}_\lambda} f^k \zeta_\lambda \, d\mathbf{x}dtd\lambda \underset{k\nearrow\infty}{\longrightarrow} \int\limits_{\Pi\times\mathbb{R}_\lambda} f\zeta_\lambda \, d\mathbf{x}dtd\lambda. \tag{15}$$

Representation (11) yields the equality

$$\int\limits_{\Pi\times\mathbb{R}_\lambda} f^k(\mathbf{x},t,\lambda)\zeta_\lambda(\mathbf{x},t,\lambda)\, d\mathbf{x}dtd\lambda = -\int\limits_{\Pi} \zeta(\mathbf{x},t,u^k(\mathbf{x},t))\, d\mathbf{x}dt. \tag{16}$$

By Tartar's theorem on Young measures [16, Chapter 3, Theorem 2.3], there is a bounded weakly measurable mapping $(\mathbf{x},t) \mapsto \nu_{x,t}$ from Π to $\operatorname{Prob}(\mathbb{R}_\lambda)$ such that $\operatorname{spt}\nu_{x,t} \subset \{\lambda : |\lambda| \leq u_*\}$ and

$$\lim_{k\nearrow\infty} \int\limits_{\Pi} \zeta(\mathbf{x},t,u^k(\mathbf{x},t))\, d\mathbf{x}dt = \int\limits_{\Pi}\bigg(\int\limits_{\mathbb{R}_\lambda} \zeta(\mathbf{x},t,\lambda)\, d\nu_{x,t}(\lambda)\bigg) d\mathbf{x}dt. \tag{17}$$

The measure $\nu_{x,t}$ is defined for a.a. $\mathbf{x}$ and t and is called the *Young measure associated with* u^k.

Using the notion of the Stieltjes integral generated by the distribution function

$$g(\mathbf{x},t,\lambda) := \int\limits_{\mathbb{R}_s} 1_{\lambda\geq s} d\nu_{x,t}(\lambda),$$

we can represent the right-hand side of (17) as

$$\int\limits_{\Pi}\bigg(\int\limits_{\mathbb{R}_\lambda} \zeta(\mathbf{x},t,\lambda)\, d\nu_{x,t}(\lambda)\bigg) d\mathbf{x}dt = \int\limits_{\Pi}\bigg(\int\limits_{\mathbb{R}_\lambda} \zeta(\mathbf{x},t,\lambda) d_\lambda g(\mathbf{x},t,\lambda)\bigg) d\mathbf{x}dt, \tag{18}$$

where $d_\lambda g(\mathbf{x},t,\cdot)$ is a parametrized Stieltjes measure on $\mathbb{R}_\lambda$. By the theory of the Stieltjes integral, the following equality holds for an arbitrary function $\psi \in C_0(\mathbb{R}_\lambda)$:

$$\int\limits_{\mathbb{R}_\lambda} \psi(\lambda)\, d_\lambda g(\mathbf{x},t,\lambda) = -\int\limits_{\mathbb{R}_\lambda} \psi'(\lambda) g(\mathbf{x},t,\lambda)\, d\lambda \quad \text{for a.e. } (\mathbf{x},t) \in \Pi.$$

Applying it, we can rewrite the right-hand side of (18) in the form

$$\int\limits_{\Pi}\bigg(\int\limits_{\mathbb{R}_\lambda} \zeta d_\lambda g\bigg) d\mathbf{x}dt = -\int\limits_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda g\, d\mathbf{x}dtd\lambda. \tag{19}$$

It follows from (15)–(19) that f and g coincide for a.e. $(\mathbf{x},t,\lambda) \in \Pi\times\mathbb{R}_\lambda$. $\square$

Introduce the set

$$\mathscr{E} := \big\{\lambda_0 \in \mathbb{R} \mid f(\cdot,\cdot,\lambda) \to f(\cdot,\cdot,\lambda_0) \text{ strongly in } L^1_{\text{loc}}(\Pi) \text{ as } \lambda \to \lambda_0\big\}.$$

From [9, Lemma 4] and Panov's theorem on modification of Tartar H-measures [9, Theorem 3] we immediately obtain the following two assertions:

Lemma 2. *The complement of $\mathscr{E}$ in $\mathbb{R}$ is at most countable, and the limit relation $f^k(\cdot,\cdot,\lambda) \underset{k\nearrow\infty}{\longrightarrow} f(\cdot,\cdot,\lambda)$ weakly* in $L^\infty(\Pi)$ holds for every $\lambda \in \mathscr{E}$.*

Theorem N. *There exist a family of locally finite Radon measures $\{\mu^{pq}\}_{p,q\in\mathscr{E}}$ on $\Pi \times \mathbb{S}^d$ and a subsequence of $\{f^k(\lambda)-f(\lambda)\}$, $\lambda \in \mathscr{E}$, such that the following equality holds for arbitrary $\Phi_1, \Phi_2 \in C_0(\Pi)$ and $\psi \in C(\mathbb{S}^d)$:*

$$\int\limits_{\Pi\times\mathbb{S}^d} \Phi_1(\mathbf{x},t)\overline{\Phi_2(\mathbf{x},t)}\psi(\mathbf{y})\,d\mu^{pq}(\mathbf{x},t,\mathbf{y})$$

$$= \lim_{k\nearrow\infty} \int\limits_{\mathbb{R}^{d+1}} \mathscr{F}[\Phi_1(f^k(p)-f(p))](\xi)\overline{\mathscr{F}[\Phi_2(f^k(q)-f(q))](\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi \quad \forall\, p,q \in \mathscr{E}. \tag{20}$$

In the statement of Theorem N and below we denote by $\overline{\varphi}$ the complex conjugate of φ. Denote by $\mathscr{F}$ the Fourier transform in $\mathbf{x}$ and t:

$$\mathscr{F}[\varphi](\xi) = \int\limits_{\mathbb{R}^{d+1}} \varphi(\mathbf{x},t)e^{2\pi i(\xi_0 t+\xi_1 x_1+\dots+\xi_d x_d)}\,d\mathbf{x}dt$$

for every integrable function φ. We assume that if a function is defined originally only for $t \in [0,T]$ then it is also defined beyond $[0,T]$ and is identically zero there. Also, sometimes we use the convenient notation $x_0 := t$.

DEFINITION 3. The family of measures $\{\mu^{pq}\}_{p,q\in\mathscr{E}}$ is called the *H-measure associated with the subsequence* $\{f^k - f\}$.

According to the general theory of H-measures, we have the following properties:

Lemma 3. 1. *For every finite set $E := \{p_1,\dots,p_n\} \subset \mathscr{E}$, the set of measures $(\mu^{p_ip_j})_{i,j=1,\dots,n}$ is Hermitian nonnegative; i.e.,*

$$\mu^{p_ip_j} = \bar{\mu}^{p_jp_i}, \quad \langle \mu^{p_ip_j}, \Phi_i\overline{\Phi}_j\psi\rangle \ge 0 \tag{21}$$

for arbitrary $\Phi_1,\dots,\Phi_n \in C_0(\Pi)$ and $\psi \in C(\mathbb{S}^d)$, $\psi \ge 0$ [14, Corollary 1.2].

2. *The mapping $(p,q) \mapsto \mu^{pq}$ is continuous from $\mathscr{E}\times\mathscr{E}$ to $\mathbb{M}(\Pi\times\mathbb{S}^d)$* [9, *Theorem* 3].

3. *For arbitrary $p,q \in \mathscr{E}$, the measure μ^{pq} is absolutely continuous with respect to the Lebesgue measure on Π. As a functional on $C(\Pi\times\mathbb{S}^d)$, it admits a natural extension to $L^2(\Pi, C(\mathbb{S}^d))$; therefore, the decomposition $d\mu^{pq}(\mathbf{x},t,\mathbf{y}) = d\sigma^{pq}_{x,t}(\mathbf{y})d\mathbf{x}dt$ holds. Here the mapping $(\mathbf{x},t) \mapsto \sigma^{pq}_{x,t}$ belongs to $L^2_w(\Pi,\mathbb{M}(S^d))$ and is determined uniquely from μ^{pq}* [17, 18].

4. *$f^k(\cdot,\cdot,\lambda) \underset{k\nearrow\infty}{\to} f(\cdot,\cdot,\lambda)$ strongly in $L^2_{\mathrm{loc}}(\Pi)$ for $\lambda \in \mathscr{E}$ if and only if $\mu^{\lambda\lambda} \equiv 0$* [14].

In Section 3 $L^2_w(\Pi,\mathbb{M}(\mathbb{S}^d))$ stands for the space of mappings $\mathbf{x} \mapsto \sigma_x$ from Π to $\mathbb{M}(\mathbb{S}^d)$ weakly measurable with respect to the Lebesgue measure on Π with the norm

$$\|\sigma\|_{L^2_w(\Pi,\mathbb{M}(\mathbb{S}^d))} = \left(\int\limits_{\Pi} \|\sigma_{x,t}\|^2_{\mathbb{M}(\mathbb{S}^d)}\,d\mathbf{x}dt\right)^{1/2} \quad \forall \sigma \in L^2_w(\Pi,\mathbb{M}(\mathbb{S}^d)).$$

§ 4. Statement of the Localization Principle for *H*-Measures

Theorem 3. *Suppose that the matrix A of the coefficients of the second derivatives of* (1a) *satisfies one of the following two conditions:* (1) *the rank d_0 of the matrix is constant or* (2) *the matrix is diagonal; i.e., $A = \operatorname{diag}(a_{11}(\mathbf{x},t),\dots,a_{dd}(\mathbf{x},t))$.*

Then the H-measure $\mu^{\lambda\lambda}$ associated with the subsequence $\{f^k - f\}$ satisfies the integral equality

$$\int\limits_{\mathbb{R}_\lambda}\Bigg(\int\limits_{\Pi\times\mathbb{S}^d}\sum_{i,j=1}^{d} b'(\lambda)a_{ij}(\mathbf{x},t)y_iy_j\zeta(\mathbf{x},t,\lambda,\mathbf{y})\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})\Bigg)d\lambda = 0 \tag{22}$$

for every function $\zeta \in C_0\big(\Pi\times\mathbb{R}_\lambda; C\big(\mathbb{S}^d_y\big)\big)$ and the integral equality

$$\int\limits_{\mathbb{R}_\lambda}\Bigg(\int\limits_{\Pi\times\mathbb{S}^d}\Bigg(y_0+\sum_{i,j=1}^{d}(a_{i\lambda}(\mathbf{x},t,\lambda)+\frac{1}{2}b'(\lambda)a_{ijx_j}(\mathbf{x},t,\lambda))y_i\Bigg)\beta(\mathbf{x},t,\lambda,\mathbf{y})\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})\Bigg)d\lambda = 0 \tag{23}$$

for every function $\beta \in C_0(\Pi\times\mathbb{R}_\lambda; C(\mathbb{S}^d_y))$ which is odd in $\mathbf{y}$; i.e., $\beta(\mathbf{x},t,\lambda,-\mathbf{y}) = -\beta(\mathbf{x},t,\lambda,\mathbf{y})$.

Corollary 1 (the localization principle). *The support of the H-measure $\mu^{\lambda\lambda}$ for a.e. $\lambda\in\mathbb{R}$ belongs to the intersection of*

$$\Bigg\{(\mathbf{x},t,\mathbf{y})\in\Pi\times\mathbb{S}^d \mid \sum_{i,j=1}^{d} b'(\lambda)a_{ij}(\mathbf{x},t)y_iy_j = 0\Bigg\}$$

and

$$\Bigg\{(\mathbf{x},t,\mathbf{y})\in\Pi\times\mathbb{S}^d \mid y_0+\sum_{i,j=1}^{d}(a_{i\lambda}(\mathbf{x},t,\lambda)+\frac{1}{2}b'(\lambda)a_{ijx_j}(\mathbf{x},t,\lambda))y_i = 0\Bigg\}.$$

Proof of Corollary 1. By the arbitrariness of ζ and nonnegativity of $\sum_{i,j=1}^{d} b'(\lambda)a_{ij}(\mathbf{x},t)y_iy_j$ and $\mu^{\lambda\lambda}$, it follows firstly from (22) that $\mu^{\lambda\lambda}$ is supported in

$$\Bigg\{(\mathbf{x},t,\mathbf{y})\in\Pi\times\mathbb{S}^d \mid \sum_{i,j=1}^{d} b'(\lambda)a_{ij}(\mathbf{x},t)y_iy_j = 0\Bigg\}.$$

By the assertion 3 of Lemma 3 and condition (2), we can secondly weaken the smoothness requirement on the test function β in (23) in the variables $\mathbf{x}$ and t, namely it suffices to require β to be compactly supported on Π (for all λ and $\mathbf{y}$) and belong to the class $L^4\big(\Pi; C_0\big(\mathbb{R}_\lambda; C\big(\mathbb{S}^d_y\big)\big)\big)$. In this connection, we can take β to be the function (odd in $\mathbf{y}$)

$$\beta(\mathbf{x},t,\lambda,\mathbf{y}) = \Bigg(y_0+\sum_{i,j=1}^{d}(a_{i\lambda}(\mathbf{x},t,\lambda)+\frac{1}{2}b'(\lambda)a_{ijx_j}(\mathbf{x},t,\lambda))y_i\Bigg)\beta_1^2(\mathbf{x},t,\lambda), \tag{24}$$

where $\beta_1 \in C_0(\Pi\times\mathbb{R}_\lambda)$ is arbitrary. Thus, from (23) we derive the integral equality

$$\int\limits_{\mathbb{R}_\lambda}\Bigg(\int\limits_{\Pi\times\mathbb{S}^d}\Bigg(y_0+\sum_{i,j=1}^{d}(a_{i\lambda}(\mathbf{x},t,\lambda)+\frac{1}{2}b'(\lambda)a_{ijx_j}(\mathbf{x},t,\lambda))y_i\Bigg)^2\beta_1^2(\mathbf{x},t,\lambda)\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})\Bigg)d\lambda = 0$$

which, by nonnegativity of the integrand and $\mu^{\lambda\lambda}$, implies that $\mu^{\lambda\lambda}$ is supported in

$$\Bigg\{(\mathbf{x},t,\mathbf{y})\in\Pi\times\mathbb{S}^d \mid y_0+\sum_{i,j=1}^{d}(a_{i\lambda}(\mathbf{x},t,\lambda)+\frac{1}{2}b'(\lambda)a_{ijx_j}(\mathbf{x},t,\lambda))y_i = 0\Bigg\}.\quad\square$$

§ 5. Proof of Theorem 3. Part I: Preliminaries

We start the proof with justification of the following auxiliary lemma:

Lemma 4. *There is a Borel measure $H \in \mathbb{M}(\Pi\times\mathbb{R}_\lambda)$ supported in $\mathbb{I}_* = \{(\mathbf{x}, t, \lambda) \in \Pi\times\mathbb{R}_\lambda : |\lambda| \le u_*\}$ and such that the limit relation*

$$m^k + b'(\lambda)n^k \to H \text{ weakly}^* \text{ in } \mathbb{M}(\Pi \times \mathbb{R}_\lambda) \quad \text{as } k \nearrow \infty \tag{25}$$

holds for an appropriate choice of a subsequence $\{k\} \subset \mathbb{N}$.

Proof. By (9a), (12), and (13), we have the uniform estimate

$$\|m^k + b'n^k\|_{(C^2(\Pi\times\mathbb{R}_\lambda))^*} \le c_* \tag{26}$$

in which the constant c_* is independent of $k \in \mathbb{N}$. Since the measure $m^k + b'n^k$ is nonnegative for an arbitrary $k \in \mathbb{N}$, we conclude that this measure has a unique natural extension to $\mathbb{M}(\Pi\times\mathbb{R}_\lambda)$ and that the set $\{m^k+b'n^k\}_{k\in\mathbb{N}}$ is uniformly bounded in the norm of $\mathbb{M}(\Pi\times\mathbb{R}_\lambda)$ by a constant c_* [19, Chapter III, § 3, Proposition 2]. It follows from this estimate that the limit relation (25) is valid for some subsequence $k \in \mathbb{N}$. The support of the measure H lies entirely in $\mathbb{I}_*$, since the supports of the measures m^k and n^k lie in $\mathbb{I}_*$ for all $k \in \mathbb{N}$. □

Alongside this lemma, in the proof of Theorem 3 we systematically use the theory of Riesz potentials and zero-order pseudodifferential operators (p.d.o.), in particular the Riesz transform.

Recall [20, Chapter 5, § 1] that, for every function $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$, the Riesz potential $\mathscr{I}_\alpha$ $(0 < \alpha < d+1)$ is defined by the formula

$$\mathscr{F}[\mathscr{I}_\alpha[\varphi]](\xi) = (2\pi|\xi|)^{-\alpha}\mathscr{F}[\varphi](\xi).$$

The Hardy–Littlewood–Sobolev Theorem [20, Chapter 5, § 1] claims that the Riesz potentials are defined on $L^p(\mathbb{R}^{d+1})$ for an arbitrary $p \in (1, +\infty)$ and present bounded mappings from $L^p(\mathbb{R}^{d+1})$ to $L^q(\mathbb{R}^{d+1})$ for $q^{-1} = p^{-1} - \alpha(d+1)^{-1}$; i.e.,

$$\|\mathscr{I}_\alpha[\varphi]\|_{L^q(\mathbb{R}^{d+1})} \le c_{p,q}\|\varphi\|_{L^p(\mathbb{R}^{d+1})} \quad \forall\varphi \in L^p(\mathbb{R}^{d+1}). \tag{27}$$

A zero-order p.d.o. $\mathscr{A}$ with symbol $\psi \in C(\mathbb{S}^d)$ is defined by the formula

$$\mathscr{F}[\mathscr{A}[\varphi]](\xi) = \psi(\xi/|\xi|)\mathscr{F}[\varphi](\xi)$$

for a function $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$. The zero-order p.d.o. $\mathscr{R}_j$ $(j = 0,\dots,d)$ with symbol $-i\xi_j/|\xi|$ is called the Riesz transform [20, Chapter 3]. The zero-order p.d.o.'s are defined and bounded on $L^p(\mathbb{R}^{d+1})$ for an arbitrary $p \in (1, +\infty)$; moreover, the following estimate is valid [20, Chapter 3, Theorem 3]:

$$\|\mathscr{A}[\varphi]\|_{L^p(\mathbb{R}^{d+1})} \le c_p\|\varphi\|_{L^p(\mathbb{R}^{d+1})} \quad \forall\varphi \in L^p(\mathbb{R}^{d+1}). \tag{28}$$

By the theory of p.d.o., the Riesz potentials and zero-order p.d.o.'s commute with each other with the operators of differentiation and satisfy the following equalities for all admissible functions (for example, for $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$):

$$(\mathscr{I}_\alpha \circ \mathscr{I}_\beta)[\varphi] = \mathscr{I}_{\alpha+\beta}[\varphi] \quad \forall\, \alpha, \beta, \alpha+\beta \in (1, d+1), \tag{29}$$

$$\mathscr{I}_1[\partial_{x_j}\varphi] = \mathscr{R}_j[\varphi], \quad j = 0,\dots,d,\ x_0 := t. \tag{30}$$

Also, note that if the symbol of the zero-order p.d.o. $\mathscr{A} : L^2(\mathbb{R}^{d+1}) \mapsto L_2(\mathbb{R}^{d+1})$ is an odd function, i.e., $\psi(-\xi/|\xi|) = -\psi(\xi/|\xi|)$, then $\mathscr{A}$ is an antiselfadjoint operator; i.e., the formula

$$\int\limits_{\mathbb{R}^{d+1}_{x,t}} \varphi_1 \cdot \mathscr{A}[\varphi_2]\, d\mathbf{x}dt = -\int\limits_{\mathbb{R}^{d+1}_{x,t}} \mathscr{A}[\varphi_1] \cdot \varphi_2\, d\mathbf{x}dt \tag{31}$$

is valid for arbitrary $\varphi_1, \varphi_2 \in L^2\big(\mathbb{R}^{d+1}_{x,t}\big)$.

The Sobolev Embedding Theorem and the above properties of the Riesz potentials yield the following assertion [20, Chapter 5, Theorem 2]:

Lemma 5. *If $p > d+1$ then the Riesz potential $\mathscr{I}_1$ is a compact operator from $L^p_{\text{loc}}(\mathbb{R}^{d+1})$ to $C_{\text{loc}}(\mathbb{R}^{d+1})$. If $1 < p \le d+1$ then the Riesz potential $\mathscr{I}_1$ is a compact operator from $L^p_{\text{loc}}(\mathbb{R}^{d+1})$ to $L^q_{\text{loc}}(\mathbb{R}^{d+1})$ for an arbitrary $q \in [1, p(d+1)(d+1-p)^{-1})$.*

Note that, applying the Plancherel Theorem to (20), we can equivalently define the H-measure $\{\mu^{pq}\}$ by the formula

$$\int\limits_{\Pi\times\mathbb{S}^d} \Phi_1\overline{\Phi}_2\psi\, d\mu^{pq}(\mathbf{x},t,\mathbf{y}) = \lim_{k\nearrow\infty}\int\limits_{\Pi} \Phi_1(f^k(p)-f(p))\overline{\mathscr{A}[\Phi_2(f^k(q)-f(q))]}\, d\mathbf{x}dt, \tag{32}$$

where $\mathscr{A}$ is the zero-order p.d.o. in $\mathbb{R}^{d+1}$ with symbol ψ.

§ 6. Proof of Theorem 3. Part II: Derivation of (22)

Denote $U^\lambda_k(\mathbf{x},t) := f^k(\mathbf{x},t,\lambda) - f(\mathbf{x},t,\lambda)$. By the limit relations (12), (13), and (25), from (10) we derive the equality

$$\int\limits_{\Pi\times\mathbb{R}_\lambda} (\zeta_t + a_{i\lambda}(\mathbf{x},t,\lambda)\zeta_{x_i} - a_{ix_i}(\mathbf{x},t,\lambda)\zeta_\lambda + b'(\lambda)\partial_{x_i}(a_{ij}(\mathbf{x},t)\zeta_{x_j}))U^\lambda_k\, d\mathbf{x}dtd\lambda + \int\limits_{\Pi\times\mathbb{R}_\lambda}\zeta_\lambda\, dH_k = 0, \tag{33}$$

where $H_k := m^k + b'(\lambda)n^k - H$ and $\zeta \in C^2_0(\Pi\times\mathbb{R}_\lambda)$ is an arbitrary test function.

Multiply the above equality by $\int_{\mathbb{R}_p}\zeta_0(p)\,dp$, where the function $\zeta_0 \in C^2_0(\mathbb{R})$ is arbitrary. Since the linear span of the set $\{\zeta(\mathbf{x},t,\lambda)\zeta_0(p)\}$ is dense in $C^2_0\big(\Pi\times\mathbb{R}^2_{\lambda,p}\big)$, from (33) we obtain the equality

$$\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} (\zeta_t + a_{i\lambda}(\mathbf{x},t,\lambda)\zeta_{x_i} - a_{ix_i}(\mathbf{x},t,\lambda)\zeta_\lambda + b'(\lambda)\partial_{x_i}(a_{ij}(\mathbf{x},t)\zeta_{x_j}))U^\lambda_k\, d\mathbf{x}dtd\lambda dp + \int\limits_{\mathbb{R}_p}\int\limits_{\Pi\times\mathbb{R}_\lambda}\zeta_\lambda\, dH_k dp = 0, \tag{34}$$

where $\zeta = \zeta(\mathbf{x},t,\lambda,p)$ is a smooth compactly-supported test function.

The further justification of (22) is based on some special choice of the test functions in (34) and the passage to the limit as $k \nearrow +\infty$.

Take ζ in the form

$$\zeta(\mathbf{x},t,\lambda,p) = \zeta_1(\mathbf{x},t)\zeta_2(\lambda)(\mathscr{I}_2\circ\mathscr{A})\big[\zeta_3(\cdot,\cdot,p)U^p_k\big](\mathbf{x},t), \tag{35}$$

where $\zeta_1 \in C^2_0(\Pi)$, $\zeta_2 \in C^2_0(\mathbb{R})$, and $\zeta_3 \in C^2_0(\Pi\times\mathbb{R}_p)$ are arbitrary and $\mathscr{A}$ is a zero-order p.d.o. with an arbitrary symbol $\psi \in C^1(\mathbb{S}^d)$. By the properties of p.d.o.'s in § 5, such choice of the test function is admissible, since all integrals in (34) are defined correctly.

Applying (29) and (30), we arrive at the equality

$$\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \big(\zeta_{1t}\zeta_2(\mathscr{I}_2\circ\mathscr{A})\big[\zeta_3U^p_k\big] + \zeta_1\zeta_2(\mathscr{I}_1\circ\mathscr{A}\circ\mathscr{R}_0)\big[\zeta_3U^p_k\big] + a_{i\lambda}(\mathbf{x},t,\lambda)\zeta_{1x_i}\zeta_2(\mathscr{I}_2\circ\mathscr{A})\big[\zeta_3U^p_k\big]$$
$$+a_{i\lambda}(\mathbf{x},t,\lambda)\zeta_1\zeta_2(\mathscr{I}_1\circ\mathscr{A}\circ\mathscr{R}_i)\big[\zeta_3U^p_k\big] - a_{ix_i}(\mathbf{x},t,\lambda)\zeta_1\zeta_{2\lambda}(\mathscr{I}_2\circ\mathscr{A})\big[\zeta_3U^p_k\big]$$
$$+b'(\lambda)a_{ijx_i}(\mathbf{x},t)\zeta_{1x_j}\zeta_2(\mathscr{I}_2\circ\mathscr{A})\big[\zeta_3U^p_k\big] + b'(\lambda)a_{ijx_i}(\mathbf{x},t)\zeta_1\zeta_2(\mathscr{I}_1\circ\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_3U^p_k\big]$$

$$+b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_{1x_ix_j}\zeta_2(\mathscr{I}_2\circ\mathscr{A})\big[\zeta_3U_k^p\big]+2b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_{1x_i}\zeta_2(\mathscr{I}_1\circ\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_3U_k^p\big]$$
$$+b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_1\zeta_2(\mathscr{A}\circ\mathscr{R}_i\circ\mathscr{R}_j)\big[\zeta_3U_k^p\big]\big)U_k^\lambda\,d\mathbf{x}dtd\lambda dp$$
$$+\int\limits_{\mathbb{R}_p}\int\limits_{\Pi\times\mathbb{R}_\lambda}\zeta_1\zeta_{2\lambda}(\mathscr{I}_2\circ\mathscr{A})\big[\zeta_3U_k^p\big]\,dH_k(\mathbf{x},t,\lambda)dp=0. \tag{36}$$

By Lemma 2, $U_k^p \underset{k\nearrow\infty}{\longrightarrow} 0$ weakly* in $L^\infty(\Pi)$ for an arbitrary $p\in\mathscr{E}$. Using this limit relation, applying the Lebesgue Dominated Convergence Theorem, Lemmas 4 and 5, and dropping down to a subsequence of $\{k\}\subset\mathbb{N}$, if need be, we arrive at the equality

$$\int\limits_{\mathbb{R}^2_{\lambda,p}}\lim_{k\nearrow+\infty}\int\limits_{\Pi}b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_1\zeta_2(\mathscr{A}\circ\mathscr{R}_i\circ\mathscr{R}_j)\big[\zeta_3U_k^p\big]U_k^\lambda\,d\mathbf{x}dtd\lambda dp=0. \tag{37}$$

By Theorem N and the fact that $\mathscr{A}\circ\mathscr{R}_i\circ\mathscr{R}_j$ is the zero-order p.d.o. with symbol $-\psi(\mathbf{y})y_iy_j$ ($\mathbf{y}\in\mathbb{S}^d$), from (37) we conclude that the following equality holds for arbitrary functions ζ_1, ζ_2, ζ_3, and ψ defined in (35):

$$\int\limits_{\mathbb{R}^2_{\lambda,p}}\int\limits_{\Pi\times\mathbb{S}^d}b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_1(\mathbf{x},t)\zeta_2(\lambda)\zeta_3(\mathbf{x},t,p)\psi(\mathbf{y})y_iy_j\,d\mu^{p\lambda}(\mathbf{x},t,\mathbf{y})\,d\lambda dp=0. \tag{38}$$

Note that the linear span of the set $\{\zeta_4(\mathbf{x},t,\lambda,p)=\zeta_2(\lambda)\zeta_3(\mathbf{x},t,p)\}$ is dense in $C_0^2\big(\Pi\times\mathbb{R}^2_{\lambda,p}\big)$ and take the test function $\zeta_4\in C_0^2\big(\Pi\times\mathbb{R}^2_{\lambda,p}\big)$ to be the Kruzhkov function [10]:

$$\zeta_4^\varepsilon(\mathbf{x},t,\lambda,p):=\frac{1}{\varepsilon}\zeta_5(\mathbf{x},t)\zeta_6\left(\frac{\lambda-p}{\varepsilon}\right)\zeta_7\left(\frac{\lambda+p}{2}\right),\quad \varepsilon>0, \tag{39}$$

where ζ_5 is a smooth compactly-supported function in Π, ζ_6 is a nonnegative even smooth function compactly supported in the interval $[-1,1]$ and having the unit mean value, i.e., $\int\zeta_6(\lambda)\,d\lambda=1$, and ζ_7 is a smooth compactly-supported function on $\mathbb{R}$.

Changing the variable p with $\kappa=(p-\lambda)/\varepsilon$, from (38) we derive

$$\int\limits_{\mathbb{R}^2_{\lambda,\kappa}}\int\limits_{\Pi\times\mathbb{S}^d}b'(\lambda)a_{ij}\zeta_1\zeta_5\zeta_6(\kappa)\zeta_7\left(\frac{2\lambda+\kappa\varepsilon}{2}\right)\psi(\mathbf{y})y_iy_jd\mu^{\lambda(\lambda+\kappa\varepsilon)}(\mathbf{x},t,\mathbf{y})\,d\lambda d\kappa=0. \tag{40}$$

By the choice of the test functions ζ_6 and ζ_7, Lemma 2, the assertion 2 of Lemma 3, and the Lebesgue Dominated Convergence Theorem, passing to the limit as $\varepsilon\searrow 0$, from (40) we infer

$$\int\limits_{\mathbb{R}_\lambda}\int\limits_{\Pi\times\mathbb{S}^d}b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_1(\mathbf{x},t)\zeta_5(\mathbf{x},t)\zeta_7(\lambda)\psi(\mathbf{y})y_iy_j\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda=0, \tag{41}$$

whence, recalling that ζ_1, ζ_5, ζ_7, and ψ are arbitrary, we immediately obtain (22).

§ 7. Proof of Theorem 3. Part III: Derivation of (23)

Introduce the regularizing kernel $\omega\in C_0^\infty(\mathbb{R})$ having the same properties as the function ζ_6 defined in the previous section. Denote

$$\omega_h(\mathbf{x}):=\frac{1}{h^d}\omega\left(\frac{x_1}{h}\right)\dots\omega\left(\frac{x_d}{h}\right),\quad (\dots)_h:=(\dots)*\omega_h,$$

and

$$U_{k,h}^p(\mathbf{x},t):=\big(U_k^p*\omega_h\big)(\mathbf{x},t)=\int\limits_{\mathbb{R}^d}\omega_h(\mathbf{x}-\tilde{\mathbf{x}})U_k^p(\tilde{\mathbf{x}},t)\,d\tilde{\mathbf{x}}.$$

Derivation of (23) is based on some special choice of the test functions in the integral equality (34). Take the test function in (34) to be the function of the form

$$\zeta(\mathbf{x},t,\lambda,p)=b'(p)\big(\zeta_1(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_{k,h}\big]\big)*\omega_h, \tag{42}$$

where $\zeta_1=\zeta_1(\mathbf{x},t,p,\lambda)$ and $\zeta_2=\zeta_2(\mathbf{x},t)$ are arbitrary smooth compactly-supported functions such that ζ_1 is symmetric in the variables λ and p, i.e., $\zeta_1(\mathbf{x},t,\lambda,p)=\zeta_1(\mathbf{x},t,p,\lambda)$ for arbitrary λ and p, and $\mathscr{A}$ is an arbitrary zero-order p.d.o. whose symbol $\psi\in C^1(\mathbb{S}^d)$ is an odd function.

Since $U^p_{k,h}$ is infinitely smooth in $\mathbf{x}$, the chosen function ζ is an admissible test function for the integral equality (34). Inserting it in (34) and using (29), (30), and the property $\langle\varphi_{1h},\varphi_2\rangle=\langle\varphi_1,\varphi_{2h}\rangle$, we derive the equality

$$\begin{aligned}\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} & b'(p)\big(U^\lambda_{k,h}\zeta_{1t}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_{k,h}\big]+U^\lambda_{k,h}\zeta_1(\mathscr{A}\circ\mathscr{R}_0)\big[\zeta_2U^p_{k,h}\big]\\
&+\big(U^\lambda_k a_{i\lambda}\big)_h\zeta_{1x_i}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_{k,h}\big]+\big(U^\lambda_k a_{i\lambda}\big)_h\zeta_1(\mathscr{A}\circ\mathscr{R}_i)\big[\zeta_2U^p_{k,h}\big]\\
&-\big(U^\lambda_k a_{ix_i}\big)_h\zeta_{1\lambda}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_k\big]+b'(\lambda)\big(U^\lambda_k a_{ijx_i}\big)_h\zeta_{1x_j}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_{k,h}\big]\\
&+b'(\lambda)\big(U^\lambda_k a_{ijx_i}\big)_h\zeta_1(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2U^p_{k,h}\big]+b'(\lambda)\big(U^\lambda_k a_{ij}\big)_h\zeta_1\partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2U^p_{k,h}\big]\\
&+2b'(\lambda)\big(U^\lambda_k a_{ij}\big)_h\zeta_{1x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2U^p_{k,h}\big]+b'(\lambda)(U^\lambda_k a_{ij})_h\zeta_{1x_ix_j}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_{k,h}\big]\big)\,d\mathbf{x}dtd\lambda dp\\
&+\int\limits_{\mathbb{R}_p}\int\limits_{\Pi\times\mathbb{R}_\lambda} b'(p)\big(\zeta_{1\lambda}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_{k,h}\big]\big)_h\,dH_k(\mathbf{x},t,\lambda)dp=0.\end{aligned} \tag{43}$$

In all integrals in (43), but the one having the form

$$\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} b'(p)b'(\lambda)\big(U^\lambda_k a_{ij}\big)_h\zeta_1\partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp, \tag{44}$$

the passage to the limit as $h\searrow 0$ is plain, for $U^p_{k,h}\underset{h\searrow0}{\to}U^p_k$ strongly in $L^1_{\mathrm{loc}}(\Pi)$. The corresponding limit expression has the form

$$\begin{aligned}\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} & b'(p)\big(U^\lambda_k\zeta_{1t}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_k\big]+U^\lambda_k\zeta_1(\mathscr{A}\circ\mathscr{R}_0)\big[\zeta_2U^p_k\big]\\
&+U^\lambda_k a_{i\lambda}\zeta_{1x_i}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_k\big]+U^\lambda_k a_{i\lambda}\zeta_1(\mathscr{A}\circ\mathscr{R}_i)\big[\zeta_2U^p_k\big]\\
&-U^\lambda_k a_{ix_i}\zeta_{1\lambda}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_k\big]+b'(\lambda)U^\lambda_k a_{ijx_i}\zeta_{1x_j}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_k\big]\\
&+b'(\lambda)U^\lambda_k a_{ijx_i}\zeta_1(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2U^p_k\big]+2b'(\lambda)U^\lambda_k a_{ij}\zeta_{1x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2U^p_k\big]\\
&+b'(\lambda)U^\lambda_k a_{ij}\zeta_{1x_ix_j}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_k\big]\big)\,d\mathbf{x}dtd\lambda dp\\
&+\int\limits_{\mathbb{R}_p}\int\limits_{\Pi\times\mathbb{R}_\lambda} b'(p)\zeta_{1\lambda}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U^p_k\big]\,dH_k(\mathbf{x},t,\lambda)dp.\end{aligned} \tag{45}$$

The passage to the limit in (44) as $h\searrow 0$ (and then as $k\nearrow+\infty$) is based on the following three lemmas:

Lemma 6. *Let $\mathscr{A}$ be a zero-order p.d.o. with symbol $\psi\in C^1(\mathbb{S}^d)$ and let $\mathscr{B}:L^2(\mathbb{R}^{d+1})\mapsto L^2(\mathbb{R}^{d+1})$ be the operator of multiplication by a function $B\in C^2_0(\mathbb{R}^{d+1}_{x,t})$; i.e., $\mathscr{B}[\varphi](\mathbf{x},t)=B(\mathbf{x},t)\varphi(\mathbf{x},t)$ $\forall\,\varphi\in L^2(\mathbb{R}^{d+1})$.*

Then the commutator $[\mathscr{A},\mathscr{B}] := \mathscr{A}\circ\mathscr{B} - \mathscr{B}\circ\mathscr{A}$ *is a continuous operator from the space* $L^2(\mathbb{R}^{d+1}_{x,t})$ *to* $W^1_2(\mathbb{R}^{d+1}_{x,t})$ *and the operator* $\varphi \mapsto \partial_{x_i}[\mathscr{A},\mathscr{B}][\varphi]$ $(i = 0,\dots,d)$ *has the structure*

$$\partial_{x_i}[\mathscr{A},\mathscr{B}][\varphi] = \sum_{j=0}^{d}(\mathscr{A}_{ij}\circ\mathscr{B}_j)[\varphi] + \mathscr{C}_i[\varphi] \quad \forall\,\varphi \in L^2(\mathbb{R}^{d+1}), \tag{46}$$

where $\mathscr{A}_{ij}$ *is the zero-order p.d.o. with symbol* $\psi_{ij} \in C(\mathbb{S}^d)$ *defined by the formula*

$$\psi_{ij}(\xi/|\xi|) = \xi_i\frac{\partial\psi(\xi/|\xi|)}{\partial\xi_j}, \quad \xi \in \mathbb{R}^{d+1}, \tag{47}$$

$\mathscr{B}_j$ *is the operator of multiplication by the function* $\partial_{x_j}B$ *(here* $x_0 := t$*), and* $\mathscr{C}_i$*:* $L^2(\mathbb{R}^{d+1}) \mapsto L^2(\mathbb{R}^{d+1})$ *is some compact operator.*

Remark 3. In terms of the variables $y_i := (\xi_i/|\xi|) \in \mathbb{S}^d$ formula (47) takes the form

$$\psi_{ij}(\mathbf{y}) = \sum_{l=0}^{d} y_i(\delta_{jl} + y_j y_l)\partial_{y_l}\psi(\mathbf{y}).$$

Lemma 6 was established in [14]. □

Lemma 7. *Let* $\zeta_1 = \zeta_1(\mathbf{x},t,p,\lambda)$ *and* $\zeta_2 = \zeta_2(\mathbf{x},t)$ *be arbitrary smooth compactly-supported functions such that* ζ_1 *is symmetric in* λ *and* p*; i.e.,* $\zeta_1(\mathbf{x},t,\lambda,p) = \zeta_1(\mathbf{x},t,p,\lambda)$ *for arbitrary* λ *and* p*, and let* $\mathscr{A}$ *be an arbitrary zero-order p.d.o. with symbol* $\psi \in C^1(\mathbb{S}^d)$ *an odd function. Then the following equality is valid:*

$$\begin{aligned}
2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} & b'(\lambda)b'(p)U^{\lambda}_{k,h}a_{ij}\zeta_1\partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2 U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&= \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^{\lambda}_{k,h}b'(\lambda)\big(a_{ij}\zeta_1\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_2]\big[b'(p)\chi U^p_{k,h}\big] \\
&+ a_{ij}\zeta_1\zeta_{2x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^p_{k,h}\big] - (a_{ij}\zeta_1)_{x_i}\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^p_{k,h}\big] \\
&- \zeta_2\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_{1ij}]\big[b'(p)\chi U^p_{k,h}\big]\big)\,d\mathbf{x}dtd\lambda dp.
\end{aligned} \tag{48}$$

Here $\mathscr{Z}_{1ij}$ and $\mathscr{Z}_2$ are the operators of multiplication by the respective functions $a_{ij}\zeta_1$ and ζ_2 and $\chi(\mathbf{x},t) = \chi^{\lambda,p}(\mathbf{x},t) = 1_{(\operatorname{supp}\zeta_1 \bigcup \operatorname{supp}\zeta_2)}(\mathbf{x},t)$. Note that $\chi\zeta_1 = \zeta_1$ and $\chi\zeta_2 = \zeta_2$.

Proof. Integrating by parts with respect to the variable x_i $(i = 1,\dots,d)$ in (44) (divided by 2) and collecting the summands so as to form explicitly the commutator of the zero-order p.d.o. $\mathscr{A}\circ\mathscr{R}_j$ and the operator of multiplication by ζ_2, we obtain the equality

$$\begin{aligned}
\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} & b'(\lambda)b'(p)U^{\lambda}_{k,h}a_{ij}\zeta_1\partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2 U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&= -\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h}b'(\lambda)a_{ij}\zeta_1\big)[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_2]\big[b'(p)\chi U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&- \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h}b'(\lambda)a_{ij}\zeta_1\big)\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp.
\end{aligned} \tag{49}$$

Integrating by parts with respect to x_i $(i = 1, \dots, d)$ the first of the integrals on the right-hand side of (49) and using the product rule for differentiation in the second integral, we establish that the following representation is valid for these two integrals:

$$
\begin{aligned}
&- \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h} b'(\lambda) a_{ij}\zeta_1\big)[\mathscr{A}\circ\mathscr{R}_j, \mathscr{Z}_2]\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&- \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h} b'(\lambda) a_{ij}\zeta_1\big)\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&= \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^{\lambda}_{k,h} b'(\lambda) a_{ij}\zeta_1 \partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j, \mathscr{Z}_2]\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&- \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h} b'(\lambda) a_{ij}\zeta_1\zeta_2\big)(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&+ \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^{\lambda}_{k,h} b'(\lambda) a_{ij}\zeta_1\zeta_{2x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp.
\end{aligned} \tag{50}
$$

Using the product rule for differentiation and collecting the summands so as to form explicitly the commutator of the zero-order p.d.o. $\mathscr{A}\circ\mathscr{R}_j$ and the operator of multiplication by the product of functions $a_{ij}\zeta_1$, we find that the second integral on the right-hand side of (50) has the representation

$$
\begin{aligned}
&- \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h} b'(\lambda) a_{ij}\zeta_1\zeta_2\big)(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&= - \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^{\lambda}_{k,h} b'(\lambda) a_{ijx_i}\zeta_1\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&- \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^{\lambda}_{k,h} b'(\lambda) a_{ij}\zeta_{1x_i}\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&+ \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h} b'(\lambda)\zeta_2\big)[\mathscr{A}\circ\mathscr{R}_j, \mathscr{Z}_{1ij}]\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&- \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h} b'(\lambda)\zeta_2\big)(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p) a_{ij}\zeta_1 U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp.
\end{aligned} \tag{51}
$$

Integrating by parts with respect to x_i $(i = 1, \dots, d)$ in the third integral on the right-hand side of (51), we find that this integral is representable as

$$
\begin{aligned}
&\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^{\lambda}_{k,h} b'(\lambda)\zeta_2\big)[\mathscr{A}\circ\mathscr{R}_j, \mathscr{Z}_{1ij}]\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp \\
&= - \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^{\lambda}_{k,h} b'(\lambda)\zeta_2 \partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j, \mathscr{Z}_{1ij}]\big[b'(p)\chi U^{p}_{k,h}\big]\, d\mathbf{x}dtd\lambda dp.
\end{aligned} \tag{52}
$$

Using (31) and the fact that the zero-order p.d.o.'s and the Riesz potentials commute with each other and with the operators of differentiation with respect to x_i, we obtain the following representation for the last integral on the right-hand side of (51):

$$
\begin{aligned}
&-\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}\big(U^\lambda_{k,h}b'(\lambda)\zeta_2\big)(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)a_{ij}\zeta_1 U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&= -\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} (\mathscr{A}\circ\mathscr{R}_j)\big[\partial_{x_i}\big(U^\lambda_{k,h}b'(\lambda)\zeta_2\big)\big]\big(b'(p)a_{ij}\zeta_1 U^p_{k,h}\big)\,d\mathbf{x}dtd\lambda dp \\
&= -\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[U^\lambda_{k,h}b'(\lambda)\zeta_2\big]\big(b'(p)a_{ij}\zeta_1 U^p_{k,h}\big)\,d\mathbf{x}dtd\lambda dp. \qquad (53)
\end{aligned}
$$

Insert (53) in the right-hand side of (51). Then insert the resulting representation in the right-hand side of (50) and the result in the right-hand side of (49).

Thus, we conclude that (44) can be represented as

$$
\begin{aligned}
&2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} b'(\lambda)b'(p)U^\lambda_{k,h}a_{ij}\zeta_1\partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2 U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&= 2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_2]\big[b'(p)\chi U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&+2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1\zeta_{2x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&-2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ijx_i}\zeta_1\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&-2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_{1x_i}\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&-2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)\zeta_2\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_{1ij}]\big[b'(p)\chi U^p_{k,h}\big]\,d\mathbf{x}dtd\lambda dp \\
&-2\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^p_{k,h}b'(p)a_{ij}\zeta_1\partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[b'(\lambda)\zeta_2 U^\lambda_{k,h}\big]\,d\mathbf{x}dtd\lambda dp. \qquad (54)
\end{aligned}
$$

Changing the notation of the variables p and λ in the last integral on the right-hand side of (54), note that this integral and (44) coincide, since ζ_1 is symmetric in p and λ.

Thus, from (54) we immediately obtain (48). □

Lemma 8. *For every $p < +\infty$ the following limit relation is valid:*

$$
\partial_{x_i}\big(\big(U^\lambda_k a_{ij}\big)_h - U^\lambda_{k,h}a_{ij}\big) \underset{h\searrow 0}{\longrightarrow} 0 \quad \text{strongly in } L^p_{\mathrm{loc}}(\Pi). \qquad (55)
$$

Proof. The result of the lemma is immediate from [21, Lemma II.1]. □

We turn to studying the passage to the limit in (44) as $h \searrow 0$. From Lemma 7 we obtain

$$\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} 2b'(p)b'(\lambda)\big(U_k^\lambda a_{ij}\big)_h \zeta_1 \partial_{x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2 U_{k,h}^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$= -\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} 2b'(p)b'(\lambda)\partial_{x_i}\big(\big(U_k^\lambda a_{ij}\big)_h - U_{k,h}^\lambda a_{ij}\big)\zeta_1(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2 U_{k,h}^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$+\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)a_{ij}\zeta_1\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_2]\big[b'(p)\chi U_{k,h}^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$+\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)a_{ij}\zeta_1\zeta_{2x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U_{k,h}^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$-\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)(a_{ij}\zeta_1)_{x_i}\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U_{k,h}^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$-\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)\zeta_2\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_{1ij}]\big[b'(p)\chi U_{k,h}^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$\stackrel{\text{def}}{=} I_{1h}+I_{2h}+I_{3h}+I_{4h}+I_{5h}. \tag{56}$$

By Lemma 8, the integral I_{1h} vanishes as $h \searrow 0$. From here and Lemma 6 we derive the limit relation

$$I_{1h}+I_{2h}+I_{3h}+I_{4h}+I_{5h}$$
$$\xrightarrow[h\searrow 0]{} \int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)a_{ij}\zeta_1\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_2]\big[b'(p)\chi U_k^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$+\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)a_{ij}\zeta_1\zeta_{2x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U_k^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$-\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)(a_{ij}\zeta_1)_{x_i}\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[b'(p)\chi U_k^p\big]\,d\mathbf{x}dtd\lambda dp$$
$$-\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)\zeta_2\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_{1ij}]\big[b'(p)\chi U_k^p\big]\,d\mathbf{x}dtd\lambda dp. \tag{57}$$

Thus, in view of (56) and the limit relations (45) and (57), letting $h \searrow 0$, from (43) we obtain the integral equality

$$\int\limits_{\Pi\times\mathbb{R}^2_{\lambda,p}} b'(p)\big(U_k^\lambda\zeta_{1t}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2 U_k^p\big] + U_k^\lambda\zeta_1(\mathscr{A}\circ\mathscr{R}_0)\big[\zeta_2 U_k^p\big] + U_k^\lambda a_{i\lambda}\zeta_{1x_i}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2 U_k^p\big]$$
$$+U_k^\lambda a_{i\lambda}\zeta_1(\mathscr{A}\circ\mathscr{R}_i)\big[\zeta_2 U_k^p\big] - U_k^\lambda a_{ix_i}\zeta_{1\lambda}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2 U_k^p\big]$$
$$+b'(\lambda)U_k^\lambda a_{ijx_i}\zeta_{1x_j}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2 U_k^p\big] + b'(\lambda)U_k^\lambda a_{ijx_i}\zeta_1(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2 U_k^p\big]$$
$$+(1/2)b'(\lambda)U_k^\lambda a_{ij}\zeta_1\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_2]\big[\chi U_k^p\big] + (1/2)b'(\lambda)U_k^\lambda a_{ij}\zeta_1\zeta_{2x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\chi U_k^p\big]$$

$$-(1/2)b'(\lambda)U_k^\lambda(a_{ij}\zeta_1)_{x_i}\zeta_2(\mathscr{A}\circ\mathscr{R}_j)\big[\chi U_k^p\big]-(1/2)b'(\lambda)U_k^\lambda\zeta_2\partial_{x_i}[\mathscr{A}\circ\mathscr{R}_j,\mathscr{Z}_{1ij}]\big[\chi U_k^p\big]$$
$$+2b'(\lambda)U_k^\lambda a_{ij}\zeta_{1x_i}(\mathscr{A}\circ\mathscr{R}_j)\big[\zeta_2U_k^p\big]+b'(\lambda)U_k^\lambda a_{ij}\zeta_{1x_ix_j}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U_k^p\big]\big)\,d\mathbf{x}dtd\lambda dp$$
$$+\int\limits_{\mathbb{R}_p}\int\limits_{\Pi\times\mathbb{R}_\lambda}b'(p)\zeta_{1\lambda}(\mathscr{I}_1\circ\mathscr{A})\big[\zeta_2U_k^p\big]\,dH_k(\mathbf{x},t,\lambda)dp=0. \tag{58}$$

Using Theorem N, Lemmas 4–6, and the Lebesgue Dominated Convergence Theorem, we pass to the limit as $k\nearrow+\infty$ in (58) (dropping down to an appropriate subsequence $\{k\}\subset\mathbb{N}$, if need be):

$$\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}^2_{\lambda,p}}b'(p)(\zeta_1(\mathbf{x},t,\lambda,p)\zeta_2(\mathbf{x},t)\psi(\mathbf{y})y_0+a_{i\lambda}(\mathbf{x},t,\lambda)\zeta_1(\mathbf{x},t,\lambda,p)\zeta_2(\mathbf{x},t)\psi(\mathbf{y})y_i$$
$$+(1/2)b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_1(\mathbf{x},t,\lambda,p)\left(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y})\right)y_iy_j\zeta_{2x_r}(\mathbf{x},t)$$
$$+2b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_1(\mathbf{x},t,\lambda,p)\zeta_{2x_i}(\mathbf{x},t)\psi(\mathbf{y})y_j$$
$$-(1/2)b'(\lambda)a_{ijx_r}(\mathbf{x},t)\zeta_1(\mathbf{x},t,\lambda,p)(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y}))y_iy_j\zeta_2(\mathbf{x},t)$$
$$-(1/2)b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_{1x_r}(\mathbf{x},t,\lambda,p)\left(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y})\right)y_iy_j\zeta_2(\mathbf{x},t)$$
$$+(1/2)b'(\lambda)a_{ijx_j}(\mathbf{x},t)\zeta_1(\mathbf{x},t,\lambda,p)\zeta_2(\mathbf{x},t)\psi(\mathbf{y})y_i)\,d\mu^{\lambda p}(\mathbf{x},t,\mathbf{y})d\lambda dp=0. \tag{59}$$

Here and in the sequel the summation over r is carried out from $r=0$ to $r=d$; moreover, $x_0:=t$.

Choosing the test function $\zeta_2=\zeta_{2N}$ so as to have $\|\zeta_{2N}\|_{C^1(\Pi)}\le c$ and $\zeta_{2N}\to1$ a.e. in Π as $N\nearrow+\infty$ and using the Lebesgue Dominated Convergence Theorem, we conclude that the function $\zeta_2\equiv1$ is an admissible test function for the integral equality (59). Thus, from (59) we obtain the equality

$$\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}^2_{\lambda,p}}b'(p)(\zeta_1(\mathbf{x},t,\lambda,p)\psi(\mathbf{y})y_0+a_{i\lambda}(\mathbf{x},t,\lambda)\zeta_1(\mathbf{x},t,\lambda,p)\psi(\mathbf{y})y_i$$
$$-(1/2)b'(\lambda)a_{ijx_r}(\mathbf{x},t)\zeta_1(\mathbf{x},t,\lambda,p)(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y}))y_iy_j$$
$$-(1/2)b'(\lambda)a_{ij}(\mathbf{x},t)\zeta_{1x_r}(\mathbf{x},t,\lambda,p)(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y}))y_iy_j$$
$$+(1/2)b'(\lambda)a_{ijx_j}(\mathbf{x},t)\zeta_1(\mathbf{x},t,\lambda,p)\psi(\mathbf{y})y_i)\,d\mu^{\lambda p}(\mathbf{x},t,\mathbf{y})d\lambda dp=0. \tag{60}$$

We take the test function ζ_1 in the form (39) which is admissible, since such function is symmetric in λ and p. Change the variable p in (60) with the variable $\kappa=(p-\lambda)/\varepsilon$ and pass to the limit as $\varepsilon\searrow0$, repeating the arguments carried out in § 6 in the derivation of the integral equality (41). Thus, from (60) we deduce the equality

$$\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda}b'(\lambda)\zeta_5\zeta_7(\lambda)\psi(\mathbf{y})(y_0+(a_{i\lambda}+(1/2)b'(\lambda)a_{ijx_j})y_i)\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda$$
$$-\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda}(b'(\lambda))^2\zeta_{5x_r}\zeta_7(\lambda)a_{ij}(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y}))y_iy_j\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda$$
$$-\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda}(b'(\lambda))^2\zeta_5\zeta_7(\lambda)a_{ijx_r}(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y}))y_iy_j\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda=0, \tag{61}$$

where $\zeta_5\in C_0^1(\Pi)$, $\zeta_7\in C_0(\mathbb{R})$, and $\psi\in C^1(\mathbb{S}^d)$ are arbitrary test functions and ψ is odd.

The second integral in (61) vanishes by (22).

For the third integral in (61) we have

Lemma 9. *The following equality is valid:*

$$\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda} (b'(\lambda))^2\zeta_5(\mathbf{x},t)\zeta_7(\lambda)a_{ijx_r}(\mathbf{x},t)(\psi_{y_r}(\mathbf{y}) + y_r y_l\psi_{y_l}(\mathbf{y}) + y_r\psi(\mathbf{y}))y_i y_j\, d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda = 0 \tag{62}$$

for arbitrary $\zeta_5 \in C_0^1(\Pi)$, $\zeta_7 \in C_0(\mathbb{R})$, *and* $\psi \in C^1(\mathbb{S}^d)$.

Proof. First consider the case when A is a diagonal matrix

$$A(\mathbf{x},t) = \mathrm{diag}(a_{11}(\mathbf{x},t), a_{22}(\mathbf{x},t),\dots,a_{dd}(\mathbf{x},t)), \quad a_{ii}(\mathbf{x},t)\ge 0\ (i=1,\dots,d).$$

In this case (62) has the form

$$\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda} (b'(\lambda))^2\zeta_5(\mathbf{x},t)\zeta_7(\lambda)\sum_{i=1}^d a_{iix_r}(\mathbf{x},t)y_i^2(\psi_{y_r}(\mathbf{y}) + y_r y_l\psi_{y_l}(\mathbf{y}) + y_r\psi(\mathbf{y}))\, d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda = 0. \tag{63}$$

Recall that, by Corollary 1, the measure $\mu^{\lambda\lambda}$ is supported in

$$M_1 = \left\{(\mathbf{x},t,\mathbf{y})\in\Pi\times\mathbb{S}^d \mid \sum_{i=1}^d a_{ii}(\mathbf{x},t)y_i^2 = 0\right\}$$

for a.e. $\lambda\in\mathbb{R}$. Hence, we conclude that (63) is equivalent to the equality

$$\int\limits_{\mathbb{R}_\lambda}\sum_{r=0}^d\int\limits_{M_1\cap M_2^r}(b'(\lambda))^2\zeta_5(\mathbf{x},t)\zeta_7(\lambda) \times\sum_{i=1}^d a_{iix_r}(\mathbf{x},t)y_i^2\left(\psi_{y_r}(\mathbf{y}) + y_r y_l\psi_{y_l}(\mathbf{y}) + y_r\psi(\mathbf{y})\right) d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda = 0, \tag{64}$$

where

$$M_2^r = \left\{(\mathbf{x},t,\mathbf{y})\in\Pi\times\mathbb{S}^d \mid \sum_{i=1}^d a_{iix_r}(\mathbf{x},t)y_i^2 \ne 0\right\}, \quad r=0,1,\dots,d.$$

Given $i=1,\dots,d$ and $r=0,1,\dots,d$, consider the sets

$$\begin{aligned} M_{2i}^r &:= \{(\mathbf{x},t,\mathbf{y})\in\Pi\times\mathbb{S}^d \mid a_{iix_r}(\mathbf{x},t)\ne 0,\ y_i\ne 0\} \\ &\equiv (\{(\mathbf{x},t)\in\Pi \mid a_{iix_r}(\mathbf{x},t)\ne 0\}\times\{\mathbf{y}\in\mathbb{S}^d \mid y_i\ne 0\}). \end{aligned}$$

The set M_2^r is a subset of $\bigcup_{i=1}^d M_{2i}^r$, since the inequality $a_{iix_r}(\mathbf{x},t)y_i^2\ne 0$ at some point $(\mathbf{x},t)\in\Pi$ is possible only if there is at least one value $i\in[1,\dots,d]$ such that $a_{iix_r}\ne 0$ and $y_i^2\ne 0$. Thus,

$$\left(M_1\cap M_2^r\right)\subset\left(M_1\cap\left(\bigcup_{i=1}^d M_{2i}^r\right)\right).$$

Consider $M_1\cap M_{2i}^r$, $i=1,\dots,d$, $r=0,\dots,d$. We have $y_i^2\ne 0$ and $a_{iix_r}(\mathbf{x},t)\ne 0$, but $a_{ii}(\mathbf{x},t)y_i^2 = 0$; hence, $a_{ii}(\mathbf{x},t) = 0$. So, $M_1\cap M_{2i}^r = \{(\mathbf{x},t)\in\Pi \mid a_{ii}(\mathbf{x},t) = 0,\ a_{iix_r}\ne 0\}\times\{\mathbf{y}\in\mathbb{S}^d \mid y_i\ne 0\}$. The set

$\{(\mathbf{x},t)\in\Pi \mid a_{ii}(\mathbf{x},t)=0,\ a_{iix_r}\neq 0\}$ is obviously a set of Lebesgue measure zero. It follows from here and the embedding

$$\big(M_1\cap M_2^r\big)\subset\left(M_1\cap\left(\bigcup_{i=1}^{d} M_{2i}^r\right)\right)=\left(\bigcup_{i=1}^{d}\Big(M_1\cap M_{2i}^r\Big)\right)$$

that $M_1\cap M_2^r$ is a subset of $\mathscr{M}^r\times\mathbb{S}^d$, where $\mathscr{M}^r=\bigcup_{i=1}^d\{(\mathbf{x},t)\in\Pi\mid a_{ii}(\mathbf{x},t)=0,\ a_{iix_r}\neq 0\}$ is a set of measure zero.

Thus, the integration with the respect to the H-measure $\mu^{\lambda\lambda}$ in (64) is carried out over a set lying in the direct product of the unit sphere $\mathbb{S}^d$ and a set of Lebesgue measure zero on Π. From here and the fact that, by the assertion 3 of Lemma 3, the H-measure $\mu^{\lambda\lambda}$ is absolutely continuous with respect to the Lebesgue measure on Π, we obtain (64). The proof of the assertion of Lemma 9 in the case of a diagonal matrix A is complete.

We turn to proving the assertion of the lemma in the case of a nondiagonal matrix with constant rank d_0. Since the matrix A is symmetric, nonnegative definite, and twice differentiable (as a function of $(\mathbf{x},t)\mapsto A(\mathbf{x},t)$), there is an orthogonal $d\times d$-matrix $Q(\mathbf{x},t)=(\theta_{ij}(\mathbf{x},t))$ which transforms A to diagonal form for arbitrary $(\mathbf{x},t)\in\Pi$: $A(\mathbf{x},t)=Q^*(\mathbf{x},t)G(\mathbf{x},t)Q(\mathbf{x},t)$, $G(\mathbf{x},t)=\operatorname{diag}(g_{11}(\mathbf{x},t),\dots,g_{d_0d_0}(\mathbf{x},t),0,\dots,0)$, and $g_{ii}>0$ $(i=1,\dots,d_0)$. Moreover, since $a_{ij}\in C^2_{\mathrm{loc}}(\Pi)$, the theorem on the differential properties of families of symmetric operators [22, Chapter II, Theorem 6.8] implies that we can choose Q so as to have

$$\theta_{ij}\in C^2_{\mathrm{loc}}(\Pi),\quad g_{ii}\in C^2_{\mathrm{loc}}(\Pi),\quad i,j=1,\dots,d. \tag{65}$$

In the integral equality (22) and the third integral in (61), for all $(\mathbf{x},t)\in\Pi$ we denote

$$Y_0=y_0,\quad Y_i=\sum_{j=1}^{d}\theta_{ij}y_j\quad(i=1,\dots,d). \tag{66}$$

By above the operator $\widetilde{\Theta}:\mathbf{y}\mapsto\mathbf{Y}$ defined by (66) is a unitary operator in $\mathbb{R}^{d+1}$ for all $(\mathbf{x},t)\in\Pi$. The family $\{\widetilde{\Theta}(\mathbf{x},t)\}$ is twice continuously differentiable on Π. Hence, the vector $\mathbf{Y}=(Y_0,\dots,Y_d)$ belongs to $\mathbb{S}^d$ for all $(\mathbf{x},t)\in\Pi$ and $\mathbf{y}\in\mathbb{S}^d$. Moreover, $\mathbf{Y}$ depends smoothly on $\mathbf{x}$ and t.

Substituting $\mathbf{Y}$ for $\mathbf{y}$ and using the representation of the H-measure $\mu^{\lambda\lambda}$ in the form $d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})=d\sigma^{\lambda\lambda}_{x,t}(\mathbf{y})d\mathbf{x}dt$ (see the assertion 3 of Lemma 3), we conclude that (22) is representable as

$$\int_{\mathbb{R}_\lambda}\left(\int_{\Pi\times\mathbb{S}^d}\sum_{i}^{d_0}b'(\lambda)g_{ii}(\mathbf{x},t)Y_i^2\tilde{\zeta}(\mathbf{x},t,\lambda,\mathbf{Y})\,d\tilde{\sigma}^{\lambda\lambda}_{x,t}(\mathbf{Y})d\mathbf{x}dt\right)d\lambda=0, \tag{67}$$

where $\tilde{\zeta}(\mathbf{x},t,\lambda,\mathbf{Y})=\zeta(\mathbf{x},t,\lambda,\widetilde{\Theta}^*(\mathbf{x},t)\mathbf{Y})$ and $\tilde{\sigma}^{\lambda\lambda}_{x,t}(\mathbf{Y})=\sigma^{\lambda\lambda}_{x,t}(\widetilde{\Theta}^*(\mathbf{x},t)\mathbf{Y})$. Since $\zeta\in C_0\big(\Pi\times\mathbb{R}_\lambda;C\big(\mathbb{S}^d_Y\big)\big)$ is arbitrary, $\tilde{\zeta}$ is arbitrary as well and belongs to the class $C_0\big(\Pi\times\mathbb{R}_\lambda;C\big(\mathbb{S}^d_Y\big)\big)$. By the above properties of the mapping $\sigma^{\lambda\lambda}_{x,t}$ and the operator $\widetilde{\Theta}$, the measure-valued mapping $\tilde{\sigma}^{\lambda\lambda}$ is nonnegative and belongs to the class $L^2_w\big(\Pi,\mathbb{M}\big(\mathbb{S}^d_Y\big)\big)$. Thus, in terms of the vector $\mathbf{Y}$, the integral equality (22) has the same form as in case A is a diagonal matrix.

Introducing the notation (66) in the third integral of (61), we arrive at the equality

$$\int_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda}(b'(\lambda))^2\zeta_5\zeta_7(\lambda)a_{ijx_r}\left(\psi_{y_r}(\mathbf{y})+y_ry_l\psi_{y_l}(\mathbf{y})+y_r\psi(\mathbf{y})\right)y_iy_j\,d\mu^{\lambda\lambda}(\mathbf{x},t,\mathbf{y})d\lambda$$

$$=\int_{\mathbb{R}_\lambda}\int_{\Pi}\int_{\mathbb{S}^d}(b'(\lambda))^2\zeta_5\zeta_7(\lambda)\sum_{i=1}^{d_0}\Big(g_{iix_r}Y_i^2+g_{ii}Y_iY_{ix_r}\Big)\Psi_r(\mathbf{Y})\,d\tilde{\sigma}^{\lambda\lambda}_{x,t}(\mathbf{Y})d\mathbf{x}dtd\lambda, \tag{68}$$

where $\Psi_r(\mathbf{Y})$ is the expression obtained from $\psi_{y_r}(\mathbf{y}) + y_r y_l \psi_{y_l}(\mathbf{y}) + y_r\psi(\mathbf{y})$ after substitution (66).

By (67), the H-measure $d\tilde{\mu}^{\lambda\lambda}(\mathbf{x}, t, \mathbf{Y}) = d\tilde{\sigma}^{\lambda\lambda}_{x,t}(\mathbf{Y})d\mathbf{x}dt$ is supported in the domain

$$M_1 = \left\{(\mathbf{x}, t, \mathbf{Y}) \in \Pi \times \mathbb{S}^d \mid \sum_{i=1}^{d_0} g_{ii}(\mathbf{x}, t)Y_i^2 = 0\right\}$$

for a.e. $\lambda \in \mathbb{R}$. Consequently, (68) is equal to the sum of the integrals

$$\int\limits_{\mathbb{R}_\lambda}\int\limits_{M_1} (b'(\lambda))^2\zeta_5\zeta_7(\lambda)\sum_{i=1}^{d_0} g_{iix_r}Y_i^2\Psi_r(\mathbf{Y})\, d\tilde{\sigma}^{\lambda\lambda}_{x,t}(\mathbf{Y})d\mathbf{x}dtd\lambda$$
$$+\int\limits_{\mathbb{R}_\lambda}\int\limits_{M_1} (b'(\lambda))^2\zeta_5\zeta_7(\lambda)\sum_{i=1}^{d_0} g_{ii}Y_iY_{ix_r}\Psi_r(\mathbf{Y})\, d\tilde{\sigma}^{\lambda\lambda}_{x,t}(\mathbf{Y})d\mathbf{x}dtd\lambda. \tag{69}$$

Note that the values Y_i are equal to zero on the set M_1 for $i = 1, \dots, d_0$, for g_{ii} is positive. Hence, both integrals in (69) and the third integral in (61) vanish. The lemma is proven. □

By Lemma 9, it follows from (61) that

$$\int\limits_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda} b'(\lambda)\zeta_5(\mathbf{x}, t)\zeta_7(\lambda)\psi(y_0 + (a_{i\lambda}(\mathbf{x}, t, \lambda)$$
$$+(1/2)b'(\lambda)a_{ijx_j}(\mathbf{x}, t))y_i)d\mu^{\lambda\lambda}(\mathbf{x}, t, \mathbf{y})d\lambda = 0 \tag{70}$$

which implies (23) in view of the arbitrariness of ζ_5, ζ_7, and ψ and the oddness of ψ. Theorem 3 is proven.

§ 8. Proof of Theorem 2

It follows from Condition G and Corollary 1 to Theorem 3 that the H-measure $\mu^{\lambda\lambda}$ is identically zero for a.e. $\lambda \in \mathbb{R}$. From here and the assertion 4 of Lemma 3 we conclude that $f^k(\cdot, \cdot, \lambda) \underset{k\nearrow+\infty}{\longrightarrow} f(\cdot, \cdot, \lambda)$ strongly in $L^1_{\text{loc}}(\Pi)$ for a.e. $\lambda \in \mathbb{R}$ and almost everywhere in $\Pi \times \mathbb{R}_\lambda$. Since f^k can take only two values 0 and 1 and f is monotone nondecreasing and right continuous in λ for a.e. $(\mathbf{x}, t)$ and such that $f \equiv 0$ for $\lambda < -u_*$ and $f \equiv 1$ for $\lambda \geq u_*$, f has the form

$$f(\mathbf{x}, t, \lambda) = \begin{cases} 1 & \text{for } \lambda \geq \tilde{u}(\mathbf{x}, t), \\ 0 & \text{for } \lambda < \tilde{u}(\mathbf{x}, t) \end{cases} \tag{71}$$

with some function $\tilde{u} \in L^\infty(\Pi)$, $\|\tilde{u}\|_{L^\infty} \leq u_*$. It follows from (11) and the limit relations (12) and (13) that $\tilde{u}$ coincides with the weak limit $u = w\text{-}\lim\limits_{k\nearrow+\infty} u^k$ and $\|u^k\|_{L^2(\mathscr{Q})} \underset{k\nearrow+\infty}{\longrightarrow} \|u\|_{L^2(\mathscr{Q})}$ for every measurable set $\mathscr{Q} \subset \Pi$. Therefore, $u^k \underset{k\nearrow+\infty}{\longrightarrow} u$ strongly in $L^2_{\text{loc}}(\Pi)$ and hence in $L^1_{\text{loc}}(\Pi)$. Theorem 2 is proven.

§ 9. Proof of Theorem 1

Introduce the parabolic approximation of (1):

$$u_t + \partial_{x_i}a_i(\mathbf{x}, t, u) - \partial_{x_i}(a_{ij}(\mathbf{x}, t)\partial_{x_j}b(u)) - \varepsilon\partial^2_{x_ix_i}u = 0, \quad \varepsilon > 0, \tag{72}$$

which is closed with the initial data (1b).

The basics of the theory of second-order parabolic equations claim [1] that (72), (1b) has a unique smooth solution u_ε for every fixed $\varepsilon > 0$. The maximum principle and the first energy inequality yield the following estimate:

$$-u_* \le u_\varepsilon \le u_* \text{ a.e. in } \Pi, \quad \|A\nabla_x u_\varepsilon\|^2_{L^2(\mathscr{Q})} + \varepsilon\|\nabla_x u_\varepsilon\|^2_{L^2(\mathscr{Q})} \le c(\mathscr{Q}), \tag{73}$$

where $\mathscr{Q} \subset \Pi$ is an arbitrary bounded domain with a sufficiently smooth boundary and the constant $c(\mathscr{Q})$ is independent of ε.

Note that (72) admits a kinetic formulation of the form (9) in which

$$dm_\varepsilon(\mathbf{x}, t, \lambda) = \varepsilon \partial_{x_i} u_\varepsilon \partial_{x_i} u_\varepsilon \, d\gamma_{u_\varepsilon(x,t)}(\lambda) d\mathbf{x} dt, \tag{74}$$

and we can choose a subsequence $\varepsilon = \varepsilon_k$ such that the limit relations (12) and (13) hold for u_{ε_k} and f_{ε_k}. Carrying out the same arguments as in the proof of Theorems 2 and 3 and Corollary 1 in § 4–§ 8 for this kinetic formulation, we establish (dropping down to a subsequence of ε_k, if need be) that

$$u_{\varepsilon_k} \underset{k\nearrow+\infty}{\longrightarrow} u \text{ strongly in } L^1_{\mathrm{loc}}(\Pi). \tag{75}$$

Now, multiply both sides of (72) by $\zeta\varphi'(u)$, where $\zeta \in C^2(\Pi)$ is an arbitrary nonnegative function vanishing in a neighborhood of the plane $\{t = T\}$ and at large $|\mathbf{x}|$ and $\varphi \in C^2_{\mathrm{loc}}(\mathbb{R})$ is an arbitrary convex function, and then integrate over Π, to obtain the equality

$$\begin{aligned}
\int\limits_\Pi \big(&\zeta_t \varphi(u_\varepsilon) + \zeta_{x_i} q_i(\mathbf{x}, t, u_\varepsilon) - \zeta\varphi'(u_\varepsilon) D_{x_i} a_i(\mathbf{x}, t, u_\varepsilon) \\
&+ \zeta D_{x_i} q_i(\mathbf{x}, t, u_\varepsilon) + w(u_\varepsilon)\partial_{x_i}(a_{ij}(\mathbf{x}, t)\partial_{x_j}\zeta) \\
&- \zeta\varphi''(u_\varepsilon) b'(u_\varepsilon)(\alpha_{il}(\mathbf{x}, t)\partial_{x_i} u_\varepsilon)(\alpha_{lj}(\mathbf{x}, t)\partial_{x_j} u_\varepsilon) \\
&+ \varepsilon\varphi(u_\varepsilon)\partial^2_{x_i x_i}\zeta - \varepsilon\zeta\varphi''(u_\varepsilon)\partial_{x_i} u_\varepsilon \partial_{x_i} u_\varepsilon\big) \, d\mathbf{x} dt + \int\limits_{\mathbb{R}^d} \varphi(u_0)\zeta(\mathbf{x}, 0) \, d\mathbf{x} = 0.
\end{aligned} \tag{76}$$

By the limit relation (75), the inequality

$$\int\limits_\Pi \varepsilon\zeta\varphi''(u_\varepsilon)\partial_{x_i} u_\varepsilon \partial_{x_i} u_\varepsilon \, d\mathbf{x} dt \ge 0$$

and the well-known lower semicontinuity property

$$\begin{aligned}
\varliminf_{\varepsilon\searrow 0} &\int\limits_\Pi \zeta\varphi''(u_\varepsilon) b'(u_\varepsilon)(\alpha_{il}(\mathbf{x}, t)\partial_{x_i} u_\varepsilon)(\alpha_{lj}(\mathbf{x}, t)\partial_{x_j} u_\varepsilon) \, d\mathbf{x} dt \\
&\ge \int\limits_\Pi \zeta\varphi''(u) b'(u)(\alpha_{il}(\mathbf{x}, t)\partial_{x_i} u)(\alpha_{lj}(\mathbf{x}, t)\partial_{x_j} u) \, d\mathbf{x} dt
\end{aligned}$$

of convex functionals (for example, see [23, Chapter 1, § 1.1.3; Chapter 2, § 2.3, Proposition 2.3.2]), letting $\varepsilon_k \searrow 0$, from (76) we derive (7). Theorem 1 is proven.

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S. A. Sazhenkov
Lavrent′ev Institute of Hydrodynamics, Novosibirsk, Russia
E-mail address: `sazhenkovs@yahoo.com`