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THE TARTAR EQUATION FOR HOMOGENIZATION OF A MODEL OF THE DYNAMICS OF FINE-DISPERSION MIXTURES

S. A. Sazhenkov UDC 517.946

Introduction

We consider a mathematical model describing a nonstationary Stokes flow of a fine-dispersion mixture of viscous incompressible fluids with rapidly oscillating initial data in a bounded domain $\Omega \subset \mathbb{R}^2$ during some time interval $[0, T]$ ($T = \text{const} < \infty$). We assume that the values of the viscosity $\nu_\varepsilon(\vec{x}, t, \lambda)$ are translated along the trajectories of particles with velocity $\vec{v}_\varepsilon(\vec{x}, t, \lambda)$, where ε and λ are arbitrary positive small parameters characterizing respectively the oscillation frequencies of viscosity distributions and the velocities and amplitudes of deviations of these distributions from a constant viscosity value $a_0 > 0$ and a sufficiently smooth velocity field $\vec{v}^{(0)}(\vec{x}, t)$ determining some "steady unperturbed" flow of an "average" homogeneous viscous fluid. The existence of solutions to this model (Problem A in Subsection 1.1) for given values of ε and λ is guaranteed by the familiar facts from the theory of the Stokes and Navier–Stokes equations [1, 2].

We perform homogenization of this model, i.e., passage to the limit in the equations and boundary conditions as $\varepsilon \to 0$. Then the problem arises of finding effective characteristics of the homogeneous medium. This problem necessitates passage to the limit in the product $\nu_\varepsilon(\nabla_x \vec{v}_\varepsilon + (\nabla_x \vec{v}_\varepsilon)^{\mathrm{T}})$ merely as ν_ε and $\nabla_x \vec{v}_\varepsilon$ converge weakly* in $L_\infty(\Omega \times [0, T])$ and weakly in $L_2(\Omega \times [0, T])$ respectively. The contemporary state of the homogenization theory makes it possible to overcome such difficulties only in the case when the medium has a certain ordered microstructure: periodic, quasi-periodic, random homogeneous, etc. [3–6]. The mixture, described by solutions to Problem A, has no such structure.

In this article we propose and implement the method of approximate determination of the effective characteristics of fine-dispersion homogeneous mixtures having no ordered structure. The method bases on the employment of the notion of H-measure proposed by L. Tartar [7]: alongside the original Problem A we consider some approximate problem (Problem C in Subsection 1.5) whose solutions γ_ε and $\vec{u}_\varepsilon$ are close to the solutions ν_ε and $\vec{v}_\varepsilon$ of Problem A. By means of the H-measure corresponding to the sequence $\{\gamma_\varepsilon\}_{\varepsilon > 0}$, we determine the structure of the weak limit $w\text{-}\lim_{\varepsilon \to 0} \gamma_\varepsilon(\nabla_x \vec{u}_\varepsilon + (\nabla_x \vec{u}_\varepsilon)^{\mathrm{T}})$ approximately with enhanced accuracy. In consequence we construct a system of approximate homogeneous equations in which the H-measure is unknown and should be determined. Finally, we close the so-obtained system by supplementing it with the macroscopic (not involving ε) Tartar evolution equation (see Subsection 1.3 and formula (1.28)) whose unique solution is the H-measure. As a result, we construct a correct closed model (Model B in Subsection 1.4) whose solutions approximate the weak limits of solutions of Problem B with high accuracy and therefore describe the motion of a homogeneous mixture rather accurately.

§ 1. Statements of Problems and the Main Results

1.1. Nonstationary Stokes flow of a fine-dispersion mixture with rapidly oscillating initial data. We consider the following

Problem A. *In the space-time cylinder* $Q_T = \{(\vec{x}, t) \in \Omega \times [0, T]\}$ (Ω *is a bounded domain in* $\mathbb{R}^2$ *with smooth boundary* $\partial\Omega$ *and* $T = \text{const} > 0$), *find the velocity field* $\vec{v}(\vec{x}, t) = \{v_1(\vec{x}, t), v_2(\vec{x}, t)\}$, *the*

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viscosity $\nu = \nu(\vec{x}, t)$, *and the pressure* $p = p(\vec{x}, t)$ *satisfying the equations*

$$\partial_t \vec{v} - \operatorname{div}_x(2\nu \mathbb{D}(\vec{v})) + \nabla_x p = \vec{f}, \quad \operatorname{div}_x \vec{v} = 0, \tag{1.1}$$

$$\partial_t \nu + \vec{v} \cdot \nabla_x \nu = 0 \tag{1.2}$$

and the initial and boundary conditions

$$\vec{v}|_{\partial\Omega} = 0, \quad \vec{v}(\vec{x}, 0) = \vec{v}_{0\varepsilon}(\vec{x}, \lambda), \quad \nu(\vec{x}, 0) = \nu_{0\varepsilon}(\vec{x}, \lambda). \tag{1.3}$$

The density of the fluid is assumed to be constant (equal to unity) in the whole domain Ω; the initial velocity distribution $\vec{v}_{0\varepsilon}(\vec{x}, \lambda)$, the initial viscosity distribution $\nu_{0\varepsilon}(\vec{x}, \lambda)$, and the vector $\vec{f}$ of mass forces are given functions satisfying the conditions

$$\vec{f} \in L_2(0, T; J_0^{2\alpha}(\Omega)), \text{ where } \alpha \in (0, 1/2) \text{ is a constant}, \tag{1.4}$$

$$\vec{v}_{0\varepsilon}(\vec{x}, \lambda) = \vec{v}_0^{(0)}(\vec{x}) + \lambda \vec{v}_{0\varepsilon}^{(1)}(\vec{x}) + \lambda^2 \vec{v}_{0\varepsilon}^{(2)}(\vec{x}), \tag{1.5}$$

$$\nu_{0\varepsilon}(\vec{x}, \lambda) = a_0 + \lambda b_{0\varepsilon}(\vec{x}) + \lambda^2 c_{0\varepsilon}(\vec{x}); \tag{1.6}$$

$$\left(\|\vec{v}_0^{(0)}\|_{J_0^{1+\alpha}(\Omega)}, \|\vec{v}_{0\varepsilon}^{(1)}\|_{J(\Omega)}, \|\vec{v}_{0\varepsilon}^{(2)}\|_{J(\Omega)}\right) \le C_0, \tag{1.7}$$

$$-c_- \le b_{0\varepsilon}(\vec{x}), c_{0\varepsilon}(\vec{x}) \le c_+ \quad \text{a.e. in } \Omega, \quad \varepsilon > 0, \tag{1.8}$$

the positive constants a_0, C_0, c_-, and c_+ are independent of ε; moreover, $a_0 - 2c_- > 0$. The following limit relations hold as $\varepsilon \to 0$:

$$\vec{v}_{0\varepsilon}^{(1)} \to \vec{v}_0^{(1)}, \quad \vec{v}_{0\varepsilon}^{(2)} \to \vec{v}_0^{(2)} \text{ strongly in } J(\Omega), \tag{1.9}$$

$$b_{0\varepsilon} \to b_0 \equiv \text{const}, \quad c_{0\varepsilon} \to c_0 \quad \text{weakly}^* \text{ in } L_\infty(\Omega). \tag{1.10}$$

The small positive parameters ε and λ characterize respectively the rapid oscillations of initial data and the small amplitude of deviations of these oscillations from the smooth "unperturbed" state $\{\vec{v}_0^{(0)}(\vec{x}), \nu_0^{(0)}(\vec{x}) \equiv a_0\}$.

A solution to Problem A is understood in the sense of the following

Definition 1.1. A *weak solution* to Problem A is a pair $\vec{v}_\varepsilon(\vec{x}, t, \lambda)$, $\nu_\varepsilon(\vec{x}, t, \lambda)$ of functions such that $\nu_\varepsilon \in L_\infty(Q_T)$, $\vec{v}_\varepsilon \in L_2(0, T; J_0^1(\Omega))$, $\partial_t \vec{v}_\varepsilon \in L_2(0, T; J^{-1}(\Omega))$, and the following equalities are valid:

$$\partial_t \vec{v}_\varepsilon - \operatorname{div}_x(2\nu_\varepsilon \mathbb{D}(\vec{v}_\varepsilon)) = \vec{f}, \tag{1.11}$$

$$\partial_t \nu_\varepsilon + \vec{v}_\varepsilon \cdot \nabla_x \nu_\varepsilon = 0, \tag{1.12}$$

$$\vec{v}_\varepsilon|_{t=0} = \vec{v}_{0\varepsilon}, \quad \nu_\varepsilon|_{t=0} = \nu_{0\varepsilon}. \tag{1.13}$$

In (1.1)–(1.13) and throughout the article we use the notations

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \nabla_x \nu = \{\partial_1 \nu, \partial_2 \nu\},$$

$$\operatorname{div}_x \vec{v} = \partial_1 v_1 + \partial_2 v_2, \quad \mathbb{D}_{ij}(\vec{v}) = (\partial_i v_j + \partial_j v_i)/2, \quad \operatorname{div} A = \left\| \sum_{i=1}^{2} \partial_i A_{ij} \right\|_{j=1,2},$$

where A is a (2×2)-matrix and $\mathbb{D}_{ij}(\vec{v})$ are the components of the stress velocity tensor. The Banach spaces $J(\Omega)$ and $J_0^k(\Omega)$ ($k \in \mathbb{R}^+$) are the closures of the set of infinitely smooth solenoidal functions compactly-supported in Ω in the norms of the Lebesgue space $L_2(\Omega)$ and the Sobolev space $W_2^k(\Omega)$. We denote by $J^{-1}(\Omega)$ the dual space of $J_0^1(\Omega)$. In Definition 1.1 and throughout the article all differential equations are understood in the distributional sense. Recall that if $\vec{v}$ belongs to $L_2(0, T; J_0^1(\Omega))$ and $\partial_t \vec{v}$ belongs to $L_2(0, T; J^{-1}(\Omega))$ then we may consider that $\vec{v} \in C([0, T]; J(\Omega))$ [2, Chapter III, § 1, Lemma 1.2]. It follows from here and (1.2) that if $\nu(\vec{x}, t)$ is a component of a weak solution to Problem A then $\nu \in C([0, T], L_q(\Omega))$, where the exponent $q < +\infty$ is arbitrary [8, Chapter II, Corollary II.2]. Hence, the traces of $\vec{v}$ and ν on the surface $\{\vec{x} \in \Omega, t = 0\}$ are well defined and the equalities in (1.13) make sense.

The following proposition is validated by the familiar methods of the theory of Navier–Stokes equations [2, Chapter III, § 2.3; 1, Chapter III, § 2]:

Proposition 1.2. *There is a weak solution to Problem* A. *Moreover, the following estimates hold:*

$$\sup_{t\in[0,T]} \|\vec{v}_\varepsilon(t)\|_{J(\Omega)} + \|\vec{v}_\varepsilon\|_{L_2(0,T;J_0^1(\Omega))} + \|\partial_t\vec{v}_\varepsilon\|_{L_2(0,T;J^{-1}(\Omega))} \le C_1, \tag{1.14}$$

$$a_0 - 2c_- \le \nu_\varepsilon(\vec{x},t,\lambda) \le a_0 + 2c_+ \quad \text{a.e. in } Q_T, \tag{1.15}$$

where C_1 *is a constant independent of* ε *and* λ.

1.2. The limit average physical characteristics of motion of a homogeneous fluid. By (1.14) and (1.15) and the compactness theorem [2, Chapter III, § 2.2], for every $\lambda < 1$ we can choose a subsequence of $\{\vec{v}_\varepsilon(\vec{x},t,\lambda),\ \nu_\varepsilon(\vec{x},t,\lambda)\}_{\varepsilon>0}$ which converges as $\varepsilon \to 0$ and possesses the following properties:

$$\vec{v}_\varepsilon \to \vec{v}_* \quad \text{weakly in } L_2(0,T;J_0^1(\Omega)),\ \text{strongly in } L_2(0,T;J(\Omega)), \tag{1.16}$$

$$\partial_t\vec{v}_\varepsilon \to \partial_t\vec{v}_* \quad \text{weakly in } L_2(0,T;J^{-1}(\Omega)), \tag{1.17}$$

$$\nu_\varepsilon \to \nu_* \quad \text{weakly}^* \text{ in } L_\infty(Q_T). \tag{1.18}$$

The vector field $\vec{v}_* = \vec{v}_*(\vec{x},t,\lambda)$ and the function $\nu_* = \nu_*(\vec{x},t,\lambda)$ describe the limit conditions of motion of the fluid as the frequency of oscillations of initial data tends to infinity. In this sense $\vec{v}_*(\vec{x},t,\lambda)$ and $\nu_*(\vec{x},t,\lambda)$ are the average physical characteristics of motion of the homogeneous mixture whose initial state is determined by the distributions

$$\vec{v}_*|_{t=0} = \vec{v}_0^{(0)} + \lambda\vec{v}_0^{(1)} + \lambda^2\vec{v}_0^{(2)}, \quad \nu_*|_{t=0} = a_0 + \lambda b_0 + \lambda^2 c_0. \tag{1.19}$$

We will see below that $\vec{v}_*(\vec{x},t,\lambda)$ and $\nu_*(\vec{x},t,\lambda)$ satisfy relations of the form

$$\partial_t\vec{v}_* - \operatorname{div}_x(\mathbb{M}_* : \nabla_x\vec{v}_*) = \vec{f}, \quad \partial_t\nu_* + \vec{v}_*\cdot\nabla_x\nu_* = 0, \tag{1.20}$$

where

$$(\mathbb{M}_* : \nabla_x\vec{v}_*)_{ik} = \sum_{j,l=1}^{2} M_*^{ijkl}\partial_j v_l, \quad i,k = 1,2.$$

The coefficients $\mathbb{M}_*^{ijkl} = \mathbb{M}_*^{ijkl}(\vec{x},t,\lambda)$ are called in mechanics the components of the *effective viscosity tensor* $\mathbb{M}_*$. As was mentioned in the introduction, the explicit form of $\mathbb{M}_*$ is unknown and hence equation (1.20) and initial conditions (1.19) do not constitute a closed model for describing the dynamics of a homogeneous mixture.

The main result of this article is the construction of a well-posed closed model whose solution approximates the weak limits

$$\vec{v}_*(\vec{x},t,\lambda) = w\text{-}\lim_{\varepsilon\to0}\vec{v}_\varepsilon(\vec{x},t,\lambda), \quad \nu_*(\vec{x},t,\lambda) = w\text{-}\lim_{\varepsilon\to0}\nu_\varepsilon(\vec{x},t,\lambda)$$

with high-order accuracy. The construction of the sought model bases on an original idea by Tartar who proposed the notion of H-measure, a nonnegative Borel measure which encodes the information about the limit conditions as ε tends to zero.

1.3. Definition of the Tartar *H*-measure. The Tartar equation. Consider some sequence ρ_ε that vanishes weakly* in $L_\infty(Q_T)$. Dropping down to a subsequence if necessary, for a.e. $t \in [0,T]$ we define the mapping μ_t from the set of functions $\{a\varphi_1\varphi_2 \mid a \in C(S^1), \varphi_1,\varphi_2 \in C_0(\Omega)\}$ into $\mathbb{R}$ as follows:

$$\langle\mu_t, a\varphi_1\varphi_2\rangle = \lim_{\varepsilon\to0}\int_\Omega \varphi_1\rho_\varepsilon\mathscr{A}[\varphi_2\rho_\varepsilon]\,d\vec{x}, \tag{1.21}$$

where $\mathscr{A}: L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ is a pseudodifferential operator of zero order with a symbol $a \in C(S^1)$; S^1 is the unit circle in $\mathbb{R}^2$. Recall that in terms of the Fourier transform

$$\mathscr{F}[u](\vec{\xi}) = \int_{\mathbb{R}^2} \exp(2\pi i \vec{x} \cdot \vec{\xi}) u(\vec{x})\, d\vec{x},$$

the operator $\mathscr{A}$ is defined as $\mathscr{F}[\mathscr{A}[u]](\vec{\xi}) = a(\vec{\xi}/|\vec{\xi}|)\mathscr{F}[u](\vec{\xi})$ and, by Parseval's identity, (1.21) has the form

$$\langle \mu_t, a\varphi_1\varphi_2 \rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \mathscr{F}[\varphi_1 \rho_\varepsilon](\vec{\xi}) a(\vec{\xi}/|\vec{\xi}|) \overline{\mathscr{F}[\varphi_2 \rho_\varepsilon](\vec{\xi})}\, d\vec{\xi}. \tag{1.22}$$

Here and in the sequel, the functions defined only for $\vec{x} \in \Omega$ and standing in the integrals over the whole space $\mathbb{R}^2$ are supposed to vanish on $\mathbb{R}^2 \setminus \overline{\Omega}$.

Since the linear span of the set of functions $\{\varphi_1\varphi_2 a\}$ is dense in $C_0(\Omega \times S^1)$, by [7, § 1], (1.21) defines a nonnegative Borel measure on $\Omega \times S^1$ for a.e. $t \in [0, T]$ which is called the *H-measure associated with the subsequence $\rho_\varepsilon \to 0$*.

Remark 1.3. The notion of H-measure was proposed also in [9] but under the name "microlocal defect measure" (MDM).

It was established in [10] that the H-measure μ_t is a natural extension of some mapping $\nu_t \in L_{2,\mathrm{w}}(\Omega, M(S^1))$ to the space of Borel measures on $\Omega \times S^1$ in the sense that the following equality is valid for every function $\varphi \in L_2(\Omega, C(S^1))$ and almost every $t \in [0, T]$:

$$\langle \mu_t, \varphi \rangle = \int_{\Omega} d\vec{x} \int_{S^1} \varphi(\vec{x}, y) d\nu_{t,x}(y),$$

which can be expressed as $d\mu_t(\vec{x}, y) = d\vec{x} d\nu_{t,x}(y)$. The mapping ν, regarded as a distribution in t, $\vec{x}$, and y, belongs to $L_\infty(0, T; L_{2,w}(\Omega, M_+(S^1)))$.

Here $M_+(S^1)$ is the set of nonnegative measures in the dual space $M(S^1)$ of $C(S^1)$. The norm in $M(S^1)$ is defined for every $\tau \in M(S^1)$ by

$$\|\tau\|_{M(S^1)} = \sup_{\|f\|_{C(S^1)} \le 1} \int_{S^1} f(y)\, d\tau(y)$$

[11, Chapter III, § 1.6]. $L_{2,w}(\Omega, M_+(S^1))$ stands for the set of weakly Lebesgue measurable mappings $\vec{x} \to \tau_x$ from Ω into $M_+(S^1)$. The norm in $L_{2,w}(\Omega, M(S^1))$ is defined as follows [12, Chapter III, Definition 2.7]:

$$\|\tau\|^2_{L_{2,w}(\Omega, M(S^1))} = \int_{\Omega} \|\tau_x\|^2_{M(S^1)}\, d\vec{x}.$$

Suppose that the sequence $\{\rho_\varepsilon\}_{\varepsilon > 0}$ under consideration is a solution to the Cauchy problem for the transport equation of the form

$$\partial_t \rho_\varepsilon + \vec{w} \cdot \nabla_x \rho_\varepsilon = 0, \quad \rho_\varepsilon|_{t=0} = \rho_{0\varepsilon}, \tag{1.23}$$

where $\vec{w} = \{w_1, w_2\}$ is a given solenoidal vector field (i.e., $\operatorname{div}_x \vec{w} = 0$) of the class $C^1(Q_T)$ which is compactly-supported in Ω and $\{\rho_{0\varepsilon}\}_{\varepsilon > 0}$ is a sequence of Cauchy data which converges weakly* to zero in $L_\infty(\Omega)$ as $\varepsilon \to 0$. By analogy to [7, Theorem 3.4], we establish that the family $\{\mu_t\}$ of H-measures associated with the sequence $\{\rho_\varepsilon(\vec{x}, t)\}$ and depending on t as a parameter is a solution to the equation

$$\int_0^T \langle \mu_t, \partial_t \Phi + \{w_1\xi_1 + w_2\xi_2, \Phi\} \rangle\, dt + \langle \mu_t|_{t=0}, \Phi|_{t=0} \rangle = 0, \tag{1.24}$$

where $(t, \vec{x}, \vec{\xi}) \in [0,T]\times\Omega\times\mathbb{R}^2$, $\Phi = \Phi(t, \vec{x}, \vec{\xi}/|\vec{\xi}|)$ is an arbitrary test function in the class $C^1([0,T]\times\Omega\times S^1)$ which satisfies the condition $\Phi|_{t=T} = 0$, and $\{\vartheta, \beta\} = \nabla_\xi\vartheta \cdot \nabla_x\beta - \nabla_x\vartheta \cdot \nabla_\xi\beta$ is the Poisson bracket. Note that the differentiation of the test function Φ with respect to ξ_i, $i = 1, 2$, in (1.24) does not lead to misunderstanding, since the Poisson bracket in (1.24) has zero order in $\vec{\xi}$ and is a function of the class $C^1[0,T] \times C(\Omega \times S^1)$, i.e., it belongs to the domain of μ_t.

Parametrizing the unit circle S^1 by the angular coordinate y, $S^1 = \{y(\operatorname{mod} 2\pi)\}$, and changing the variables ξ_1 and ξ_2 by $\xi_1 = r\cos y$ and $\xi_2 = r\sin y$, with r the radial coordinate on the plane, we take (1.24) to the form

$$\int_0^T \langle \mu_t, \partial_t\Phi + \operatorname{div}_x(\Phi\vec{w}) + (Y : \nabla_x\vec{w})\partial_y\Phi\rangle\, dt + \langle \mu_t|_{t=0}, \Phi|_{t=0}\rangle = 0, \tag{1.25}$$

where

$$Y = \begin{pmatrix} -\frac{1}{2}\sin 2y & \cos^2 y \\ -\sin^2 y & \frac{1}{2}\sin 2y \end{pmatrix}$$

and $\Phi = \Phi(t, \vec{x}, y)$ is a test function such that $\Phi \in C^1([0,T] \times \Omega \times S^1)$ and $\Phi|_{t=T} = 0$.

In the distributional sense, equation (1.25) is equivalent to the first-order linear partial differential equation

$$\partial_t\mu_t + \vec{w} \cdot \nabla_x\mu_t + \partial_y(\mu_t Y : \nabla_x\vec{w}) = 0, \quad (t, \vec{x}, y) \in (0,T) \times \Omega \times S^1. \tag{1.26}$$

Definition 1.4. Equation (1.26) is called the *Tartar equation.*

The Tartar equation for an H-measure was rigorously derived in [10] under weaker assumptions on the smoothness of $\vec{w}$, namely, under $\vec{w} \in L_2(0,T; J_0^2(\Omega))$. In [13] (the results were also announced in [14]) the author proved well-posedness of the Cauchy problem for the Tartar equation in the class of nonnegative Borel measures.

Proposition 1.5. *Suppose that $\vec{w} \in L_2(0,T; J_0^1(\Omega))$. Then, for every measure μ_0 such that $d\mu_0(\vec{x}, y) = d\vec{x}d\nu_{0,x}(y)$, $\nu_0 \in L_{2,w}(\Omega, M_+(S^1))$, the Cauchy problem for (1.26) with the Cauchy data $\mu_t|_{t=0} = \mu_0$ has a unique solution; moreover, $d\mu_t(\vec{x}, y) = d\vec{x}d\nu_{t,x}(y)$, where $\nu \in L_\infty(0,T; L_{2,w}(\Omega, M_+(S^1)))$.*

1.4. Description of a closed homogeneous model.

Model B. Find successively the following:

(B1) a vector field $\vec{v}^{(0)} = \vec{v}^{(0)}(\vec{x}, t)$ solving the problem of "unperturbed" flow of an "averaged" homogeneous fluid

$$\partial_t\vec{v}^{(0)} - a_0\Delta_x\vec{v}^{(0)} = \vec{f}, \quad \vec{v}^{(0)} \in L_2(0,T; J_0^{2+\alpha}(\Omega)), \quad \vec{v}^{(0)}|_{t=0} = \vec{v}_0^{(0)}; \tag{1.27}$$

(B2) a measure μ_t such that

$$d\mu_t(\vec{x}, y) = d\vec{x}d\nu_{t,x}(y), \quad \nu \in L_\infty(0,T; L_{2,w}(\Omega, M_+(S^1))),$$

solving the Cauchy problem for the Tartar equation

$$\partial_t\mu_t + \vec{v}^{(0)} \cdot \nabla_x\mu_t + \partial_y(\mu_t Y : \nabla_x\vec{v}^{(0)}) = 0, \quad \mu_t|_{t=0} = \mu_0, \tag{1.28}$$

with μ_0 the H-measure associated with the sequence $b_{0\varepsilon} - b_0$;

(B3) a function $\gamma = \gamma(\vec{x}, t, \lambda)$, $\gamma \in L_\infty(Q_T) \cap C([0,T]; L_q(\Omega))$, $q \in [1, +\infty)$, solving the Cauchy problem for the transport equation:

$$\partial_t\gamma + \vec{v}^{(0)} \cdot \nabla_x\gamma = 0, \quad \gamma|_{t=0} = a_0 + \lambda b_0 + \lambda^2 c_0; \tag{1.29}$$

(B4) a vector field $\vec{u} = \vec{u}(\vec{x}, t, \lambda)$ solving the problem

$$\partial_t\vec{u} - \operatorname{div}_x(\Lambda\mathbb{D}(\vec{u}) + \mathbb{D}(\vec{u})\Lambda) = \vec{f}, \quad \vec{u} \in L_2(0,T; J_0^1(\Omega)), \tag{1.30}$$

$$\vec{u}|_{t=0} = \vec{v}_0^{(0)} + \lambda\vec{v}_0^{(1)} + \lambda^2\vec{v}_0^{(2)}, \tag{1.31}$$

with

$$\Lambda_{ij} = \delta_{ij}\gamma - \lambda^2 a_0^{-1} \int\limits_{S^1} Y_{1ij}\, d\nu_{t,x}(y), \quad Y_1 = \begin{pmatrix} \cos^2 y & \frac{1}{2}\sin 2y \\ \frac{1}{2}\sin 2y & \sin^2 y \end{pmatrix}$$

and δ_{ij} the Kronecker symbol.

A *solution* of Model B is the triple $\{\mu_t, \gamma, \vec{u}\}$. In the definition of the (2×2)-matrix Λ we use the representation $d\mu_t(\vec{x}, y) = d\vec{x}d\nu_{t,x}(y)$ for the H-measure μ_t. Accordingly, equation (1.30) is understood in the sense of the integral equality

$$\int\limits_{Q_T} \vec{u}\cdot\partial_t\vec{\varphi}\, d\vec{x}dt + \int\limits_{\Omega} \vec{u}(\vec{x},0)\cdot\vec{\varphi}(\vec{x},0)\, d\vec{x} - \int\limits_{Q_T} \gamma\mathbb{D}(\vec{u}) : \mathbb{D}(\vec{\varphi})\, d\vec{x}dt$$

$$+\frac{\lambda^2}{a_0}\int\limits_0^T \Big(\int\limits_{\Omega\times S^1} (Y_1\mathbb{D}(\vec{u}) + \mathbb{D}(\vec{u})Y_1) : \mathbb{D}(\vec{\varphi})\, d\mu_t(\vec{x},y)\Big)\, dt + \int\limits_{Q_T} \vec{f}\cdot\vec{\varphi}\, d\vec{x}dt = 0, \tag{1.32}$$

where $\vec{\varphi} = \vec{\varphi}(\vec{x}, t)$ is a sufficiently smooth test function such that $\vec{\varphi}|_{t=T} = 0$.

1.5. Description for an approximate model. In § 2 Model B will be constructed by homogenization in ε of the following model.

Problem C. Find successively the following:

(C1) a function $\gamma_\varepsilon = \gamma_\varepsilon(\vec{x}, t, \lambda)$, $\gamma_\varepsilon \in L_\infty(Q_T)\cap C([0,T]; L_q(\Omega))$, $q \in [1, +\infty)$, solving the Cauchy problem for the transport equation

$$\partial_t\gamma_\varepsilon + \vec{v}^{(0)}\cdot\nabla_x\gamma_\varepsilon = 0, \quad \gamma_\varepsilon|_{t=0} = a_0 + \lambda b_{0\varepsilon} + \lambda^2 c_{0\varepsilon}, \tag{1.33}$$

where $\vec{v}^{(0)}$ is a solution to (B1);

(C2) a vector field $\vec{u}_\varepsilon = \vec{u}_\varepsilon(\vec{x}, t, \lambda)$ solving the problem

$$\partial_t\vec{u}_\varepsilon - \operatorname{div}_x(2\gamma_\varepsilon\mathbb{D}(\vec{u}_\varepsilon)) = \vec{f}, \quad \vec{u}_\varepsilon \in L_2(0,T; J_0^1(\Omega)), \tag{1.34}$$

$$\vec{u}_\varepsilon|_{t=0} = \vec{v}_0^{(0)} + \lambda\vec{v}_{0\varepsilon}^{(1)} + \lambda^2\vec{v}_{0\varepsilon}^{(2)}. \tag{1.35}$$

The pair $\{\gamma_\varepsilon, \vec{u}_\varepsilon\}$ is a *solution* to Problem C.

1.6. Well-posedness and the approximation properties. Below we describe the construction of Model B and establish the following theorems:

Theorem 1.6. *Model* B *has a unique solution for all values of λ less than some $\lambda_{**} \in (0, 1)$.*

The exact value λ_{**} will be established in the proof.

Theorem 1.7. *Let $\{\vec{v}_*, \nu_*\}$ be the weak limit of solutions to Problem* A *and let $\vec{u}$ and γ, together with μ_t, constitute a solution of Model* B*. Then as $\lambda \to 0$*

$$\|\vec{v}_* - \vec{u}\|_{L_2(0,T;J_0^1(\Omega))} + \|\partial_t\vec{v}_* - \partial_t\vec{u}\|_{L_2(0,T;J^{-1}(\Omega))} = O(\lambda^2), \tag{1.36}$$

$$\|\nu_* - \gamma\|_{C([0,T];L_q(\Omega))} = o(\lambda^2), \quad q \in [1, +\infty). \tag{1.37}$$

Theorem 1.8. *Let $\{\vec{u}_*, \gamma_*\}$ be the weak limit of solutions to Problem* C *and let $\vec{u}$ and γ, together with μ_t, constitute a solution of Model* B*. Then*

$$\gamma_*(\vec{x}, t) = \gamma(\vec{x}, t) \quad \text{a.e. in } Q_T, \tag{1.38}$$

$$\|\vec{u}_* - \vec{u}\|_{L_2(0,T;J_0^1(\Omega))} + \|\partial_t\vec{u}_* - \partial_t\vec{u}\|_{L_2(0,T;J^{-1}(\Omega))} = O(\lambda^3) \quad \text{as } \lambda \to 0. \tag{1.39}$$

§ 2. Construction of Model B

2.1. Well-posedness of Problem C.

Proposition 2.1. *Problem* C *has a unique solution for arbitrary given* $\varepsilon, \lambda > 0$. *This solution satisfies* (1.14) *and* (1.15) *with* $\vec{u}_\varepsilon$ *and* γ_ε *substituted for* $\vec{v}_\varepsilon$ *and* ν_ε.

Proof. Subproblem (B1) is uniquely solvable [15, Chapter IV, § 2]. Subproblem (C1) in whose statement $\vec{v}^{(0)}$, a solution to subproblem (B1), is considered as a given function is also uniquely solvable [8, Corollary II.1]. Finally, considering a solution γ_ε to subproblem (C1) as a function given in the statement of subproblem (C2), we establish unique solvability of subproblem (C2) by arguments similar to [2, Chapter III, § 1.2–§ 1.4]. Estimates for solutions to Problem C are established by analogy with Problem A.

Remark 2.2. The above proof of solvability of subproblem (C1) also justifies solvability of subproblem (B3). In view of Propositions 1.5 and 2.1, we have thus partially established Theorem 1.6, namely, solvability of subproblems (B1)–(B3).

2.2. Averaging of Problem C. By estimates (1.14) and (1.15) and the compactness theorem of [2, Chapter III, § 2.2], from a sequence of solutions to Problem C we can extract a subsequence that converges weakly as $\varepsilon \to 0$ to some limit $\{\gamma_*, \vec{u}_*\}$ and is such that

$$\vec{u}_\varepsilon \to \vec{u}_* \quad \text{weakly in } L_2\big(0,T; J^1_0(\Omega)\big), \text{ strongly in } L_2(0,T; J(\Omega)), \tag{2.1}$$

$$\partial_t \vec{u}_\varepsilon \to \partial_t \vec{u}_* \quad \text{weakly in } L_2(0,T; J^{-1}(\Omega)), \tag{2.2}$$

$$\gamma_\varepsilon \to \gamma_* \quad \text{weakly}^* \text{ in } L_\infty(Q_T). \tag{2.3}$$

According to the theory of G-convergence of parabolic differential operators (see [16] wherein the results of [17, Theorems 13–16] concerning scalar-valued operators are extended to vector-valued operators), we also have the limit relation

$$2\gamma_\varepsilon \mathbb{D}(\vec{u}_\varepsilon) \to \mathbb{X}_* : \nabla_x \vec{u}_* \quad \text{weakly in } L_2(Q_T), \tag{2.4}$$

with $\mathbb{X}_* \in L_\infty(Q_T)$ the effective viscosity tensor satisfying the estimates

$$\big|\mathbb{X}^{ijkl}_*(\vec{x}, t, \lambda)\big| \le C_2 \quad \text{a.e. in } Q_T, \tag{2.5}$$

$$\int\limits_\Omega \sum_{i,j,k,l=1}^{2} \mathbb{X}^{ijkl}_*(\vec{x}, t, \lambda)\partial_j \varphi_l(\vec{x})\partial_i \varphi_k(\vec{x})\, d\vec{x} \ge C^2_k(\Omega)(a_0 - \lambda c_- - \lambda^2 c_-)\|\vec{\varphi}\|^2_{J^1_0(\Omega)} \tag{2.6}$$

for a.e. $t \in [0,T]$ and every $\vec{\varphi} \in J^1_0(\Omega)$, where C_2 is a constant depending only on the geometry of the domain Ω and the constants a_0, c_-, and c_+, and $C_k(\Omega)$ is the constant in Korn's inequality $\|\vec{\varphi}\|_{J^1_0(\Omega)} \le C_k(\Omega)\|\mathbb{D}(\vec{\varphi})\|_{2,\Omega}$, $\vec{\varphi} \in J^1_0(\Omega)$ [18, Chapter III, § 3.2] (here and in the sequel $\|\mathbb{D}(\vec{\varphi})\|^2_{2,\Omega} = \int_\Omega |\mathbb{D}(\vec{\varphi})|^2 d\vec{x}$, $|\mathbb{D}(\vec{\varphi})|^2 = \mathbb{D}(\vec{\varphi}) : \mathbb{D}(\vec{\varphi})$). We have thus established the following

Proposition 2.3. *The weak limits* $\{\gamma_*, \vec{u}_*\}$ *of a sequence of solutions to Problem* C *and the effective viscosity tensor* $\mathbb{X}_*$ *satisfy the equalities*

$$\partial_t \vec{u}_* - \operatorname{div}_x(\mathbb{X}_* : \nabla_x \vec{u}_*) = \vec{f}, \quad \vec{u}_*|_{t=0} = \vec{v}^{(0)}_0 + \lambda \vec{v}^{(1)}_0 + \lambda^2 \vec{v}^{(2)}_0; \tag{2.7}$$

$$\partial_t \gamma_* + \vec{v}^{(0)} \cdot \nabla_x \gamma_* = 0, \quad \gamma_*|_{t=0} = a_0 + \lambda b_0 + \lambda^2 c_0. \tag{2.8}$$

Remark 2.4. Equality (1.20) is established similarly.

2.3. A lemma on asymptotic expansions.

Lemma 2.5. *For all λ less than $\lambda_* \stackrel{\mathrm{def}}{=} (1 + 32a_0^{-1}[\max\{c_-, c_+\}]^2)^{-1/2}$, the following hold:*

(a) *The solutions to Problem* C *and the weak limits $\{\gamma_*, \vec{u}_*\}$ of a sequence of solutions to Problem* C *admit the representations*

$$\gamma_\varepsilon(\vec{x}, t, \lambda) = a_0 + \lambda\gamma_\varepsilon^{(1)}(\vec{x}, t) + \lambda^2\gamma_\varepsilon^{(2)}(\vec{x}, t), \tag{2.9}$$

$$\vec{u}_\varepsilon(\vec{x}, t, \lambda) = \vec{v}^{(0)}(\vec{x}, t) + \lambda\vec{u}_\varepsilon^{(1)}(\vec{x}, t) + \lambda^2\vec{u}_\varepsilon^{(2)}(\vec{x}, t) + \sum_{r\geq 3}\lambda^r\vec{u}_\varepsilon^{(r)}(\vec{x}, t), \tag{2.10}$$

$$\gamma_*(\vec{x}, t, \lambda) = a_0 + \lambda b_0 + \lambda^2\gamma_*^{(2)}(\vec{x}, t), \tag{2.11}$$

$$\vec{u}_*(\vec{x}, t, \lambda) = \vec{v}^{(0)}(\vec{x}, t) + \lambda\vec{u}_*^{(1)}(\vec{x}, t) + \lambda^2\vec{u}_*^{(2)}(\vec{x}, t) + \sum_{r\geq 3}\lambda^r\vec{u}_*^{(r)}(\vec{x}, t), \tag{2.12}$$

with

$$\gamma_*^{(2)} = w\text{-}\lim_{\varepsilon\to 0}\gamma_\varepsilon^{(2)}, \quad \vec{u}_*^{(r)} = w\text{-}\lim_{\varepsilon\to 0}\vec{u}_\varepsilon^{(r)}, \ r \geq 1, \tag{2.13}$$

$$-c_- \leq \gamma_\varepsilon^{(1)}(\vec{x}, t), \gamma_\varepsilon^{(2)}(\vec{x}, t) \leq c_+ \quad \text{for a.e. } (\vec{x}, t) \in Q_T, \tag{2.14}$$

$$\|\vec{u}_\varepsilon^{(r)}\|_{L_2(0,T;J_0^1(\Omega))} \leq C_3\lambda_*^{2-r}, \quad r \geq 1; \tag{2.15}$$

moreover, $\gamma_\varepsilon^{(1)}$, $\gamma_\varepsilon^{(2)}$, $\gamma_*^{(2)}$, and $\vec{u}_\varepsilon^{(r)}$ are determined uniquely and the constant C_3 depends only on C_0, a_0, c_-, c_+, and norm of the right-hand side of (1.34) *in $L_2(0, T; J^{-1}(\Omega))$;*

(b) *The effective viscosity tensor $\mathbb{X}_*$ admits the analytic representation*

$$\mathbb{X}_*(\vec{x}, t, \lambda) = \mathbb{X}_*^{(0)} + \lambda\mathbb{X}_*^{(1)} + \lambda^2\mathbb{X}_*^{(2)}(\vec{x}, t) + \sum_{r\geq 3}\lambda^r\mathbb{X}_*^{(r)}(\vec{x}, t); \tag{2.16}$$

moreover, the following relations are valid for every function $\vec{\varphi} \in L_2(0, T; J_0^1(\Omega))$:

$$\mathbb{X}_*^{(0)} : \nabla_x\vec{\varphi} = 2a_0\mathbb{D}(\vec{\varphi}), \ \mathbb{X}_*^{(1)} : \nabla_x\vec{\varphi} = 2b_0\mathbb{D}(\vec{\varphi}), \ \|\mathbb{X}_*^{(2)} : \nabla_x\vec{\varphi}\|_{2,Q_T} \leq C_4\|\nabla_x\vec{\varphi}\|_{2,Q_T}, \tag{2.17}$$

where C_4 is a constant independent of λ and the choice of $\vec{\varphi}$.

Remark 2.6. The identity in (2.17) is equivalent to the following:

$$\mathbb{X}_*^{(0)ijkl} = (\delta_{il}\delta_{jk} + \delta_{ij}\delta_{lk})a_0, \quad \mathbb{X}_*^{(1)ijkl} = (\delta_{il}\delta_{jk} + \delta_{ij}\delta_{lk})b_0, \quad i, j, k, l = 1, 2.$$

Proof of Lemma 2.5. Equalities (2.9) and (2.11), estimate (2.14), and the first limit relation in (2.13) are established by simple arguments using the linearity of the transport equation (1.33), the existence and uniqueness theorem for it, and the a priori estimates for its solutions [8; 1, Chapter III, § 1]. To justify the other assertions of Lemma 2.5, consider the following auxiliary problem:

Problem D. Find a vector-function $\vec{w}_\varepsilon = \vec{w}_\varepsilon(\vec{x}, t, \lambda)$ satisfying

$$\partial_t\vec{w}_\varepsilon - \operatorname{div}_x\big(2\big(a_0 + \lambda\gamma_\varepsilon^{(1)} + \lambda^2\gamma_\varepsilon^{(2)}\big)\mathbb{D}(\vec{w}_\varepsilon)\big) = \vec{g}, \quad \vec{w}_\varepsilon \in L_2\big(0, T; J_0^1(\Omega)\big), \tag{2.18}$$

$$\vec{w}_\varepsilon|_{t=0} = \vec{v}_0^{(0)} + \lambda\vec{v}_{0\varepsilon}^{(1)} + \lambda^2\vec{v}_{0\varepsilon}^{(2)}, \tag{2.19}$$

where $\vec{g} \in L_2(0, T; J^{-1}(\Omega))$ is given and $a_0 + \lambda\gamma_\varepsilon^{(1)} + \lambda^2\gamma_\varepsilon^{(2)}$ is a solution to subproblem (C1).

The formal asymptotic solution (see [4, Chapter II]) to Problem D

$$\vec{w}_\varepsilon^{\text{f.a.s.}}(\vec{x}, t, \lambda) = \vec{w}^{(0)}(\vec{x}, t) + \lambda\vec{w}_\varepsilon^{(1)}(\vec{x}, t) + \lambda^2\vec{w}_\varepsilon^{(2)}(\vec{x}, t) + \sum_{r\geq 3}\lambda^r\vec{w}_\varepsilon^{(r)}(\vec{x}, t), \tag{2.20}$$

constructed by successively solving the problems

$$\partial_t \vec{w}^{(0)} - a_0 \Delta_x \vec{w}^{(0)} = \vec{g}, \quad \vec{w}^{(0)} \in L_2\big(0,T; J_0^1(\Omega)\big), \tag{2.21}$$

$$\vec{w}^{(0)}|_{t=0} = \vec{v}_0^{(0)}, \tag{2.22}$$

$$\partial_t \vec{w}_\varepsilon^{(1)} - a_0 \Delta_x \vec{w}_\varepsilon^{(1)} = \operatorname{div}_x\big(2\gamma_\varepsilon^{(1)} \mathbb{D}(\vec{w}^{(0)})\big), \quad \vec{w}_\varepsilon^{(1)} \in L_2\big(0,T; J_0^1(\Omega)\big), \tag{2.23}$$

$$\vec{w}_\varepsilon^{(1)}|_{t=0} = \vec{v}_{0\varepsilon}^{(1)}, \tag{2.24}$$

$$\partial_t \vec{w}_\varepsilon^{(2)} - a_0 \Delta_x \vec{w}_\varepsilon^{(2)} = \operatorname{div}_x\big(2\gamma_\varepsilon^{(2)} \mathbb{D}(\vec{w}^{(0)}) + 2\gamma_\varepsilon^{(1)} \mathbb{D}\big(\vec{w}_\varepsilon^{(1)}\big)\big),\ \vec{w}_\varepsilon^{(2)} \in L_2\big(0,T; J_0^1(\Omega)\big), \tag{2.25}$$

$$\vec{w}_\varepsilon^{(2)}|_{t=0} = \vec{v}_{0\varepsilon}^{(2)}, \tag{2.26}$$

$$\partial_t \vec{w}_\varepsilon^{(r)} - a_0 \Delta_x \vec{w}_\varepsilon^{(r)} = \operatorname{div}_x\big(2\gamma_\varepsilon^{(2)} \mathbb{D}\big(\vec{w}_\varepsilon^{(r-2)}\big) + 2\gamma_\varepsilon^{(1)} \mathbb{D}\big(\vec{w}_\varepsilon^{(r-1)}\big)\big),\ \vec{w}_\varepsilon^{(r)} \in L_2\big(0,T; J_0^1(\Omega)\big), \tag{2.27}$$

$$\vec{w}_\varepsilon^{(r)}|_{t=0} = 0, \quad r \geq 3, \tag{2.28}$$

is an exact solution. Indeed, the functions $\vec{w}_\varepsilon^{(r)}$, $r \geq 1$, satisfy an inequality like (2.1) which is established by successively estimating $\big\|\mathbb{D}\big(\vec{w}_\varepsilon^{(1)}\big)\big\|^2_{2,Q_T}$ by $\big\|\mathbb{D}(\vec{w}^{(0)})\big\|^2_{2,Q_T}$, $\big\|\mathbb{D}\big(\vec{w}_\varepsilon^{(2)}\big)\big\|^2_{2,Q_T}$ by $\big\|\mathbb{D}\big(\vec{w}_\varepsilon^{(1)}\big)\big\|^2_{2,Q_T}, \dots,$ $\big\|\mathbb{D}\big(\vec{w}_\varepsilon^{(r)}\big)\big\|^2_{2,Q_T}$ by $\big\|\mathbb{D}\big(\vec{w}_\varepsilon^{(r-1)}\big)\big\|^2_{2,Q_T}$ on using the familiar technics of construction of a priori estimates for solutions to the Navier–Stokes equations [2, Chapter III, § 1.3] and Korn's inequality. It follows from (2.15) and Minkowki's inequality that series (2.20) converges in $L_2\big(0,T; J_0^1(\Omega)\big)$. Proposition 2.1 implies in turn that the above-constructed solution to Problem D is unique.

Subproblem (C2) is a particular case of Problem D; therefore, from (2.20) we immediately derive representation (2.10) and estimate (2.15). Passing to the limit in (2.10) as $\varepsilon \to 0$, we establish (2.12) and the second limit relation in (2.13). Thus, item (a) of Lemma 2.5 is proven.

We turn to the proof of item (b). It bases on the fact that, according to the theory of G-convergence [16, 17], the tensor $\mathbb{X}_*$ is uniquely determined by the family $\{\gamma_\varepsilon\}_{\varepsilon>0}$ and the initial data $\vec{w}_\varepsilon|_{t=0}$ and is independent of the choice of the right-hand side $\vec{g}$ of (2.18). Extract a subsequence from $\{\gamma_\varepsilon\}_{\varepsilon>0}$ (if necessary) and pass to the limit in Problem D as $\varepsilon \to 0$. According to the theory of G-convergence, we have

$$2\gamma_\varepsilon \mathbb{D}(\vec{w}_\varepsilon) \to \mathbb{X}_* : \nabla_x \vec{w}_* \text{ weakly in } L_2(Q_T), \quad \vec{w}_\varepsilon \to \vec{w}_* \text{ weakly in } L_2(0,T; J_0^1(\Omega)). \tag{2.29}$$

By item (a), the function $\vec{w}_*$ and the convolution $\mathbb{X}_* : \nabla_x \vec{w}_*$ are analytic in λ for a.e. $(\vec{x}, t) \in Q_T$. Therefore, by the arbitrariness of $\vec{g}$ we establish as in [19] analyticity of the tensor $\mathbb{X}_*(\vec{x}, t, \lambda)$, i.e., validity of a representation like $\mathbb{X}_*(\vec{x}, t, \lambda) = \sum_{r=0}^\infty \lambda^r \mathbb{X}_*^{(r)}(\vec{x}, t)$.

To complete the proof of the lemma, consider Problem D with the data

$$\vec{g} = \partial_t \vec{\varphi} - a_0 \Delta_x \vec{\varphi}, \tag{2.30}$$

where $\vec{\varphi}$ is an arbitrary function such that

$$\vec{\varphi} \in L_2\big(0,T; J_0^1(\Omega)\big), \quad \partial_t \vec{\varphi} \in L_2(0,T; J^{-1}(\Omega)), \quad \vec{\varphi}|_{t=0} = \vec{v}_0^{(0)}.$$

Note that in this case $\vec{w}^{(0)} = \vec{\varphi}$ by (2.21) and (2.22). By (2.29) and asymptotic expansions like (2.10) and (2.12), we have $\mathbb{X}_*^{(0)} : \nabla_x \vec{\varphi} = 2a_0 \mathbb{D}(\vec{\varphi})$ and $\mathbb{X}_*^{(0)} : \nabla_x \vec{w}_*^{(1)} + \mathbb{X}_*^{(1)} : \nabla_x \vec{\varphi} = 2a_0 \mathbb{D}(\vec{w}_*^{(1)}) + 2b_0 \mathbb{D}(\vec{\varphi})$. By the arbitrariness of $\vec{\varphi}$, the above equalities imply validity of the equalities in (2.17). Hence, we derive

$$\mathbb{X}_*^{(2)} : \nabla_x \vec{\varphi} = 2c_0 \mathbb{D}(\vec{\varphi}) + 2w\text{-}\lim_{\varepsilon \to 0} \gamma_\varepsilon \mathbb{D}\big(\vec{w}_\varepsilon^{(1)}\big) - 2b_0 \mathbb{D}\big(\vec{w}_*^{(1)}\big). \tag{2.31}$$

From here and the energy estimate

$$\sup_\varepsilon \big\|\mathbb{D}\big(\vec{w}_\varepsilon^{(1)}\big)\big\|^2_{2,Q_T} \leq 16 a_0^{-1} [\max\{c_-, c_+\}]^2 \|\mathbb{D}(\vec{\varphi})\|^2_{2,Q_T}$$

for a solution to problem (2.23), (2.24) (which is obtained by the above-mentioned technics of construction of estimates for solutions to Navier–Stokes equations [2, Chapter III, § 1.3]) we obtain the inequality in (2.17) with $C_4 \stackrel{\text{def}}{=} 2\max\{c_-, c_+\} + 16a_0^{-1/2}(\max\{c_-, c_+\})$.

2.4. The explicit form of $\mathbb{X}_*^{(2)}(\vec{x}, t)$. The problem of determining the explicit form of the effective viscosity tensor $\mathbb{X}_*(\vec{x}, t, \lambda)$ is beyond the scope of the methods of the theory of G-convergence which were used in Subsections 2.1 and 2.2. In this subsection we use Tartar's idea [7, § 4.2] of determining the effective coefficients of averaged equations by means of H-measures and establish the explicit form of $\mathbb{X}_*^{(2)}$ in terms of the H-measure associated with the sequence $\{\gamma_\varepsilon^{(1)} - b_0\}_{\varepsilon>0}$.

Proposition 2.7. *The following integral identity holds:*

$$\int\limits_{Q_T} (\mathbb{X}_*^{(2)} : \nabla_x \vec{\varphi}) : \Phi \, d\vec{x}dt = \int\limits_{Q_T} \gamma_*^{(2)} \mathbb{D}(\vec{\varphi}) : \Phi \, d\vec{x}dt$$

$$-\frac{1}{a_0} \int\limits_0^T \Bigg(\int\limits_{\Omega\times S^1} (Y_1 \mathbb{D}(\vec{\varphi}) + \mathbb{D}(\vec{\varphi}) Y_1) : \Phi \, d\mu_t(\vec{x}, y) \Bigg) dt, \tag{2.32}$$

where $\Phi = \Phi(\vec{x}, t)$ is a (2×2)-matrix whose entries are smooth test functions, $\vec{\varphi} = \vec{\varphi}(\vec{x}, t)$ is a sufficiently smooth test vector-function, and μ_t is the H-measure associated with $\{\gamma_\varepsilon^{(1)} - b_0\}$.

Remark 2.8. Equality (2.32) and the representation $d\mu_t(\vec{x}, y) = d\vec{x} d\nu_{t,x}(y)$ for the H-measure μ_t imply that the components of the tensor $\mathbb{X}_*^{(2)}$ have the following form a.e. in Q_T:

$$\mathbb{X}_*^{(2)ijkl}(\vec{x}, t) = (\delta_{il}\delta_{jk} + \delta_{ij}\delta_{lk})\gamma_*^{(2)}(\vec{x}, t)$$

$$-(2a_0)^{-1} \int\limits_{S^1} [\delta_{il} Y_{1jk}(y) + \delta_{kl} Y_{1ij}(y) + \delta_{ij} Y_{1kl}(y) + \delta_{jk} Y_{1il}(y)] d\nu_{t,x}(y).$$

Proof of Proposition 2.7. Consider Problem D with data of the form (2.30). As in the proof of Lemma 2.5, we denote a solution to this problem by $\vec{w}_\varepsilon$. By item (a) of Lemma 2.5, it has the form $\vec{w}_\varepsilon = \vec{\varphi} + \lambda \vec{w}_\varepsilon^{(1)} + \lambda^2 \vec{w}_\varepsilon^{(2)} + \dots$. We express the structure of the weak limit $w\text{-}\lim_{\varepsilon\to 0} \gamma_\varepsilon^{(1)} \mathbb{D}(\vec{w}_\varepsilon^{(1)})$ in terms of the H-measure μ_t associated with the sequence $\{\gamma_\varepsilon^{(1)} - b_0\}$. To this end, we apply the Fourier transform in the space variables to both sides of (2.23) and write down the integral equality equivalent in the distributional sense to the resultant relation:

$$\int\limits_0^T dt \int\limits_{\mathbb{R}^2} \mathscr{F}\big[\vec{w}_\varepsilon^{(1)}\big](\vec{\xi}, t) \cdot \partial_t \vec{\Psi}(\vec{\xi}, t) \, d\vec{\xi} - \int\limits_{\mathbb{R}^2} \mathscr{F}\big[\vec{v}_{0\varepsilon}^{(1)}\big](\vec{\xi}) \cdot \vec{\Psi}(\vec{\xi}, 0) \, d\vec{\xi}$$

$$+ \int\limits_0^T dt \int\limits_{\mathbb{R}^2} \sum_{j,k=1}^2 2\pi i \xi_j \mathscr{F}\big[2a_0 \mathbb{D}_{jk}(\vec{w}_\varepsilon^{(1)}) + 2\gamma_\varepsilon^{(1)} \mathbb{D}_{jk}(\vec{\varphi})\big](\vec{\xi}, t) \Psi_k(\vec{\xi}, t) \, d\vec{\xi} = 0, \tag{2.33}$$

where $\vec{\Psi}$: $\mathbb{R}^2 \times [0, T] \to \mathbb{R}^2$ is a sufficiently smooth vector-function and $\vec{\Psi}(\vec{\xi}, T) = 0$.

Suppose that $\vec{\psi} \in C^1(\Omega \times (0, T))$, $\vec{\psi}|_{t=T} = 0$, and $a \in C^1(S^1)$. (Still formally) take the test function in (2.33) to be $\vec{\Psi}_\varepsilon = |\vec{\xi}|^{-1} a(\vec{\xi}/|\vec{\xi}|) \overline{\mathscr{F}[(\gamma_\varepsilon^{(1)} - b_0)\vec{\psi}](\vec{\xi}, t)}$ and apply Parseval's identity (also formally). By the equality $\partial_t(\gamma_\varepsilon^{(1)} - b_0) = -\operatorname{div}_x(\vec{v}^{(0)}(\gamma_\varepsilon^{(1)} - b_0))$ which is an elementary consequence of (2.9) and (1.33), as a result we derive from (2.33) the integral identity

$$0 = 2\pi \int\limits_0^T dt \int\limits_{\mathbb{R}^2} \vec{w}_\varepsilon^{(1)} \cdot (\mathscr{I}_1 \circ \mathscr{A}) \big[\big(\gamma_\varepsilon^{(1)} - b_0\big) \partial_t \vec{\psi} \big] \, d\vec{x}$$

$$-2\pi \int\limits_0^T dt \int\limits_{\mathbb{R}^2} \sum_{j,k=1}^2 w_{\varepsilon k}^{(1)} (\mathscr{R}_j \circ \mathscr{A}) \big[v_j^{(0)} \big(\gamma_\varepsilon^{(1)} - b_0\big) \psi_k \big] \, d\vec{x}$$

$$+2\pi \int\limits_0^T dt \int\limits_{\mathbb{R}^2} \sum_{j,k=1}^2 w_{\varepsilon k}^{(1)} (\mathscr{I}_1 \circ \mathscr{A}) \big[v_j^{(0)} \big(\gamma_\varepsilon^{(1)} - b_0\big) \partial_j \psi_k \big] \, d\vec{x} - \int\limits_{\mathbb{R}^2} \vec{v}_{0\varepsilon}^{(1)} (\mathscr{I}_1 \circ \mathscr{A}) [(b_{0\varepsilon} - b_0) \vec{\psi}(0)] \, d\vec{x}$$

$$+2\pi \int\limits_0^T dt \int\limits_{\mathbb{R}^2} \sum_{j,k=1}^2 \big\{ 2a_0 \mathbb{D}_{jk}\big(\vec{w}_\varepsilon^{(1)}\big) + 2\big(\gamma_\varepsilon^{(1)} - b_0\big) \mathbb{D}_{jk}(\vec{\varphi}) \big\} (\mathscr{R}_j \circ \mathscr{A}) \big[\big(\gamma_\varepsilon^{(1)} - b_0\big) \psi_k \big] \, d\vec{x}$$

$$-2\pi \int\limits_0^T dt \int\limits_{\mathbb{R}^2} \sum_{j,k=1}^2 2b_0 \mathbb{D}_{jk}(\vec{\varphi}) (\mathscr{R}_j \circ \mathscr{A}) \big[\big(\gamma_\varepsilon^{(1)} - b_0\big) \psi_k \big] \, d\vec{x}. \tag{2.34}$$

Here $\mathscr{A}$ is the pseudodifferential operator of zero order with the symbol a, $\mathscr{R}_j$, $j = 1, 2$, is the Riesz operator (the pseudodifferential operator of zero order with the symbol $i\xi_j/|\xi|$), and $\mathscr{I}_1$ is the Riesz potential defined by $\mathscr{F}[\mathscr{I}_1[\Phi]](\vec{\xi}) = (2\pi|\vec{\xi}|)^{-1} \mathscr{F}[\Phi](\vec{\xi})$, $\Phi \in \mathscr{D}'(\mathbb{R}^2)$ [20, Chapter V, § 1.1]. By the theory of pseudodifferential operators of zero order and Riesz potentials [20, Chapter II, Theorem 3; Chapter V, Theorem 1] and the above-established regularity properties of the functions in the integrands, all integrals in (2.34) are well defined. Consequently, the choice of $\vec{\Psi}_\varepsilon$ as a test function is legitimate, the application of Parseval's identity is justified, and the integral identity (2.34) makes sense.

Letting $\varepsilon \to 0$ in (2.34), from the weak* convergence of $\gamma_\varepsilon^{(1)}$ to b_0 in $L_\infty(Q_T)$, the strong convergence of $\vec{w}_\varepsilon^{(1)}$ to $\vec{w}_*^{(1)}$ in $L_2(0, T; J(\Omega))$, and the above-mentioned properties of pseudodifferential operators we obtain the equality

$$\lim_{\varepsilon \to 0} i \int\limits_0^T dt \int\limits_{\mathbb{R}^2} \sum_{j,k=1}^2 \mathbb{D}_{jk}\big(\vec{w}_\varepsilon^{(1)}\big) (\mathscr{R}_j \circ \mathscr{A}) \big[\big(\gamma_\varepsilon^{(1)} - b_0\big) \psi_k \big] \, d\vec{x}$$

$$= \int\limits_0^T \left\langle \mu_t, \sum_{j,k=1}^2 \mathbb{D}_{jk}(\vec{\varphi}) \psi_k \frac{1}{a_0} \frac{\xi_j}{|\vec{\xi}|} a(\vec{\xi}/|\vec{\xi}|) \right\rangle dt$$

whose right-hand side is represented in terms of the H-measure μ_t associated with the chosen subsequence $\gamma_\varepsilon^{(1)} - b_0 \xrightarrow{w} 0$.

We put $\vec{\psi}_l = \mathbb{D}_{\cdot l}(\vec{\zeta})$ in this equality ($\vec{\zeta} \in C^1([0, T]; C_0^1(\Omega))$ is an arbitrary test vector-function) and $a_l(\vec{\xi}/|\vec{\xi}|) = \xi_l/|\vec{\xi}|$ (i.e., $\mathscr{A} = -i\mathscr{R}_l$), $l = 1, 2$. After simple technical transformations we conclude that

$$\lim_{\varepsilon \to 0} \int\limits_0^T dt \int\limits_\Omega \big(\gamma_\varepsilon^{(1)} - b_0\big) \mathbb{D}\big(\vec{w}_\varepsilon^{(1)}\big) : \mathbb{D}(\zeta) \, d\vec{x}$$

$$= -\frac{1}{2} \int\limits_0^T \left\langle \mu_t, \sum_{j,k,l=1}^2 \partial_j \varphi_l \partial_i \zeta_k \left\{ \frac{\xi_j \xi_k}{a_0 |\vec{\xi}|^2} \delta_{il} + \frac{\xi_l \xi_k}{a_0 |\vec{\xi}|^2} \delta_{ij} + \frac{\xi_i \xi_j}{a_0 |\vec{\xi}|^2} \delta_{kl} + \frac{\xi_i \xi_l}{a_0 |\vec{\xi}|^2} \delta_{kj} \right\} \right\rangle dt$$

$$= -\frac{1}{a_0} \int\limits_0^T \int\limits_{\Omega \times S^1} (Y_1 \mathbb{D}(\vec{\varphi}) + \mathbb{D}(\vec{\varphi}) Y_1) : \mathbb{D}(\vec{\zeta}) \, d\mu_t(\vec{x}, y) dt. \tag{2.35}$$

In the last equality, y is the angular coordinate on the unit circle. Recall that $\xi_1/|\vec{\xi}| = \cos y$ and $\xi_2/|\vec{\xi}| = \sin y$. Equalities (2.17), (2.31), and (3.35) imply the integral identity (2.32).

2.5. Discarding the summands in (2.16) that involve the factors $\mathbb{X}_*^{(r)}$, $r \ge 3$, indefinite by Lemma 2.5 and Proposition 2.7, i.e., "discarding the tail" of the analytic representation for the effective viscosity tensor $\mathbb{X}_*(\vec{x}, t, \lambda)$, we arrive at the statement of subproblem (B4) in which, by Remarks 2.6 and 2.8, the matrix Λ corresponds to the effective viscosity tensor $\tilde{\mathbb{X}}_* = \mathbb{X}_*^{(0)} + \lambda\mathbb{X}_*^{(1)} + \lambda^2\mathbb{X}_*^{(2)}$; i.e., $\tilde{\mathbb{X}}_* : \nabla_x\vec{\varphi} = \Lambda\mathbb{D}(\vec{\varphi}) + \mathbb{D}(\vec{\varphi})\Lambda$, $\vec{\varphi} \in J_0^1(\Omega)$.

§ 3. Justification of Well-Posedness and Proof of the Approximation Properties of Model B

3.1. Proof of Theorem 1.6. Recall that by Remark 2.2 subproblems (B1)–(B3) are uniquely solvable. Owing to the estimate of item (b) of Lemma 2.5, for every $\lambda < \lambda_{**} \stackrel{\text{def}}{=} (c_-^2(2C_4)^{-2} + a_0 C_4^{-1})^{-1/2} - c_-(2C_4)^{-1}$, the tensor $\widetilde{\mathbb{X}}_*$ corresponding to Λ determines for a.e. $(\vec{x}, t) \in Q_T$ the bounded positive definite quadratic form $\Phi \to (\widetilde{\mathbb{X}}_* : \Phi) : \Phi$ (Φ is the (2×2)-matrix with entries in $\mathbb{R}$). Indeed, boundedness is obvious and positive definiteness follows from the two inequalities $(\widetilde{X}_* : \Phi) : \Phi \ge (a_0 - \lambda c_- - \lambda^2 C_4)|\Phi|^2$ for a.e. $(\vec{x}, t) \in Q_T$ and $a_0 - \lambda c_- - \lambda^2 C_4 > 0$ for $\lambda < \lambda_{**}$. By analogy with [2, Chapter III, § 1.2–§ 1.4], we hence conclude that subproblem (B4) has a unique solution.

3.2. Proof of Theorem 1.8. The arguments of Subsection 2.5 imply that the function γ and the vector-function $\vec{u}$ constituting, together with the measure μ_t, a solution of Model B satisfy item (a) of Lemma 2.5; moreover, the terms of the asymptotic expansions of γ and $\vec{u}$ up to the second order in λ coincide with similar terms for the weak limits of solutions to Problem C: $\gamma^{(0)} = a_0$, $\gamma^{(1)} = b_0$, $\gamma^{(2)} = \gamma_*^{(2)}$, $\vec{u}^{(0)} = \vec{v}^{(0)}$, $\vec{u}^{(1)} = \vec{u}_*^{(1)}$, and $\vec{u}^{(2)} = \vec{u}_*^{(2)}$. Thus, identity (1.38) holds together with the representation

$$\vec{u} - \vec{u}_* = \sum_{r \ge 3} \lambda^r (\vec{u}^{(r)} - \vec{u}_*^{(r)})$$

which, in view of estimates like (1.38) for $\vec{u}^{(r)}$ and $\vec{u}_*^{(r)}$, implies the asymptotic relation (1.39).

Remark 3.1. Model B yields a better approximation to the weak limits of solutions to Problem C as compared with any other model not accounting for the information encoded in the construction of the H-measure μ_t. Indeed, denote by $\vec{u}$ a velocity field that is a solution of the model in question (say, "Model E"). Using the constructions of § 2 and § 3, we see that the maximally possible accuracy of approximation to the velocity field $\vec{u}_* = w\text{-}\lim_{\varepsilon \to 0} \vec{u}_\varepsilon$ for Model E is at most the second order in λ:

$$\|\vec{u}_* - \vec{U}\|_{L_2(0,T;J_0^1(\Omega))} + \|\partial_t\vec{u}_* - \partial_t\vec{U}\|_{L_2(0,T;J^{-1}(\Omega))} = O(\lambda^2) \\ \ne o(\lambda^2) \quad \text{as } \lambda \to 0.$$

3.3. Proof of Theorem 1.7. Repeating the arguments of the proof of Lemma 2.5, we establish the following representations for $\vec{v}_\varepsilon$, ν_*, and $\vec{v}_*$:

$$\vec{v}_\varepsilon(\vec{x}, t, \lambda) = \vec{v}^{(0)}(\vec{x}, t) + \lambda\vec{v}_\varepsilon^{(1)}(\vec{x}, t, \lambda) + \lambda^2\vec{v}_\varepsilon^{(2)}(\vec{x}, t, \lambda) + \lambda^3\vec{v}_\varepsilon^{(3)}(\vec{x}, t, \lambda) + \dots, \tag{3.1}$$

$$\nu_*(\vec{x}, t, \lambda) = a_0 + \lambda b_0 + \lambda^2\nu_*^{(2)}(\vec{x}, t, \lambda), \tag{3.2}$$

$$\vec{v}_*(\vec{x}, t, \lambda) = \vec{v}^{(0)}(\vec{x}, t) + \lambda\vec{u}_*^{(1)}(\vec{x}, t, \lambda) + \lambda^2\vec{v}_*^{(2)}(\vec{x}, t, \lambda) + \lambda^3\vec{v}_*^{(3)}(\vec{x}, t, \lambda) + \dots, \tag{3.3}$$

where $\nu_*^{(2)}$ satisfies an estimate like (2.14); $\vec{v}_*^{(2)}$ and $\vec{v}_*^{(3)}$, estimates like (2.15); and the vector field $\vec{u}_*^{(1)}$ is defined in item (a) of Lemma 2.5. Considering the asymptotic representations (2.11), (2.12), (3.2), and (3.3) and using the identities $\gamma^{(2)} = \gamma_*^{(2)}$ and $\vec{u}^{(1)} = \vec{u}_*^{(1)}$, we obtain

$$\gamma - \nu_* = \lambda^2(\gamma^{(2)} - \nu_*^{(2)}), \tag{3.4}$$

$$\vec{v}_* - \vec{u} = \lambda^2(\vec{v}_*^{(2)} - \vec{u}_*^{(2)}) + \sum_{r\ge 3} \lambda^r(\vec{v}_*^{(r)} - \vec{u}_*^{(r)}). \tag{3.5}$$

Since $\vec{v}_* \underset{\lambda\to 0}{\longrightarrow} \vec{v}^{(0)}$ strongly in $L_2(0,T;J_0^1(\Omega))$, the stability theorem for transport equations [8, Theorem II.4] yields the following limit relation:

$$\gamma^{(2)} - \nu_*^{(2)} \underset{\lambda\to 0}{\longrightarrow} 0 \quad \text{strongly in } C([0,T], L_q(\Omega)),\ q < +\infty. \tag{3.6}$$

From here and (3.4) we obtain the asymptotic relation (1.37).

Finally, the asymptotic relation (1.36) ensues from (3.5) in view of estimates like (2.15) for $\vec{v}_*^{(r)}$ and $\vec{u}_*^{(r)}$ $(r \ge 2)$.

Appendix. Equivalent Description for Model B in the Case of a Periodic Rapidly Oscillating Initial Viscosity

Closing the article, we consider a specific case of Model B in which the oscillations of the initial viscosity distribution in the statement of Problem A (Problem C) are periodic:

$$\nu_\varepsilon(\vec{x},t,\lambda)|_{t=0} = \gamma_\varepsilon(\vec{x},t,\lambda)|_{t=0} = a_0 + \lambda B(\vec{x}/\varepsilon), \tag{4.1}$$

where $B \in C^1(\mathbb{R}^2)$, $-c_- \le B(\vec{\theta}) \le c_+$, $\vec{\theta} \in \mathbb{R}^2$, and B is periodic with period 1, i.e., $B(\vec{\theta} + \vec{e}_i) = B(\vec{\theta})$, $\vec{\theta} \in \mathbb{R}^2$, $i = 1, 2$, $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$.

Given an arbitrary sequence $\varepsilon_k \underset{k\to\infty}{\longrightarrow} 0$, we have [6, Theorem 0.2]

$$\gamma_{\varepsilon_k}|_{t=0} \to a_0 + \lambda\langle B\rangle \quad \text{weakly}^* \text{ in } L_\infty(\Omega), \tag{4.2}$$

where $\langle B\rangle = \int_{(0,1)\times(0,1)} B(\vec{\theta})\, d\vec{\theta}$. A solution to the Cauchy problem for the transport equation (1.33) with initial data (4.1) has the form [1, § 3, Lemma 1.1]

$$\gamma_\varepsilon = a_0 + \lambda B(\vec{X}(\vec{x},t)/\varepsilon), \tag{4.3}$$

where $\vec{X}(\vec{x},t) = \vec{V}(\tau,\vec{x},t)|_{\tau=0}$ and $\vec{V}(\tau,\vec{x},t)$ is a solution to the Cauchy problem

$$\frac{d\vec{V}}{d\tau} = \vec{v}^{(0)}(\vec{V},\tau), \quad \vec{V}|_{\tau=t} = \vec{x}. \tag{4.4}$$

In view of (4.2), $\gamma_{\varepsilon_k} \underset{\varepsilon_k\to 0}{\longrightarrow} a_0 + \lambda\langle B\rangle$ weakly* in $L_\infty(Q_T)$. By [9, Proposition 1.5], the H-measure associated with the sequence $\{B(\vec{X}(\vec{x},t)/\varepsilon_k) - \langle B\rangle\}$ weakly convergent to zero is calculated explicitly and looks like

$$\mu_t(\vec{x},y) = \sum_{\vec{p}\in\mathbb{Z}^2\setminus\{0,0\}} |B_p|^2 \delta\left(y - \cos^{-1}\left\{\vec{p}\cdot\partial_{x_1}\vec{X} \Big/ \left(\sum_{i=1}^{2} (\vec{p}\cdot\partial_{x_i}\vec{X})^2\right)^{1/2}\right\}\right); \tag{4.5}$$

i.e., for every function $\Phi(\vec{x},y)$ of the class $L_2(\Omega, C(S^1))$ we have

$$\int_{\Omega\times S^1} \Phi(\vec{x},y)d\mu_t(\vec{x},y) = \int_\Omega \sum_{\vec{p}\in\mathbb{Z}^2\setminus\{0,0\}} |B_p|^2 \Phi\left(\vec{x}, \cos^{-1}\left\{\vec{p}\cdot\partial_{x_1}\vec{X} \Big/ \left(\sum_{i=1}^{2} (\vec{p}\cdot\partial_{x_i}\vec{X})^2\right)^{1/2}\right\}\right) d\vec{x}. \tag{4.6}$$

In (4.5) and (4.6), $\vec{p}$ is a multi-index (p_1, p_2), δ is the Dirac delta-function, and $B_p = \int_{(0,1)\times(0,1)} B(\vec{\theta})$ $e^{2\pi i\vec{p}\cdot\vec{\theta}}\, d\vec{\theta}$ are the Fourier coefficients of $B(\vec{\theta})$.

Representation (4.5) implies that, firstly, the Tartar equation in Model B can be replaced with equation (4.4) which makes it possible to extract all information about the evolution of the H-measure and, secondly, the effective coefficients Λ_{ij} in Model B can be expressed explicitly in terms of $B(\vec{\theta})$. Thus, the original Model B is reduced to a form, not involving the notion of H-measure, in which (by analogy with the classical theory of averaging of periodic structures) we have a direct connection between the shape of a weakly convergent sequence of oscillating distributions and the shape of the limit effective characteristics of a homogeneous medium.

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