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# Kinetic formulation for the Graetz–Nusselt ultra-parabolic equation ☆

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**Abstract**

The well-posedness of the Cauchy problems for a quasilinear ultra-parabolic equation with partial diffusion and discontinuous convection coefficients is established for both entropy and kinetic formulations. The kinetic formulation is set up and solved by means of studying of the Young measures, associated with sequences of solutions of parabolic approximations. The kinetic equation appears as the linear scalar equation, which describes the evolution of the distribution functions of the Young measures in time and space, and which involves an additional 'kinetic' variable. The proofs of the principal results of the paper are based on the originally constructed renormalization procedure for the kinetic equation.


*Keywords:* Ultra-parabolic equations; Non-isotropic diffusion; Kinetic formulation; Entropy and measure-valued solutions

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## 1. Problem formulation and main results

We are interested in proposing of the existence and uniqueness theory for quasilinear equations with partial diffusion and discontinuous convection coefficients. More precisely, in this paper we consider the Cauchy problem for the equation

$$\mathbb{R}^d \times (0,T): \quad \partial_t u + \operatorname{div}_x\big(\boldsymbol{v}a(u)\big) - \operatorname{div}_x\big(A\nabla_x b(u)\big) = 0, \tag{1.1a}$$

endowed with periodic initial data belonging to $L^\infty(\mathbb{R}^d)$ and periodicity conditions

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in \mathbb{R}^d, \tag{1.1b}$$

$$u(\boldsymbol{x}+\boldsymbol{e}_i,t) = u(\boldsymbol{x},t) \quad \text{for a.e. } (\boldsymbol{x},t) \in \mathbb{R}^d \times (0,T). \tag{1.1c}$$

Without loss of generality, we assume that

$$0 \leqslant u_0(\boldsymbol{x}) \leqslant 1 \quad \text{a.e. in } \mathbb{R}^d. \tag{1.2}$$

Here $\boldsymbol{e}_i$ $(i = 1,\dots,d)$ are standard basis vectors in $\mathbb{R}^d$, $u(\boldsymbol{x},t)$ is an unknown function, $A \neq 0$ is a symmetric non-negative matrix, the flux $a$ and the diffusion function $b$ satisfy the conditions

$$a \in C^1_{\mathrm{loc}}(\mathbb{R}), \qquad b \in C^2_{\mathrm{loc}}(\mathbb{R}), \qquad b'(u) > 0 \quad \text{for } u \in \mathbb{R}. \tag{1.3}$$

The velocity field $\boldsymbol{v}$ is given and we suppose that $\boldsymbol{v}$, $\nabla_x \boldsymbol{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times [0,T])$ and

$$\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{e}_i,t) = \boldsymbol{v}(\boldsymbol{x},t), \qquad \operatorname{div}_x \boldsymbol{v}(\boldsymbol{x},t) = 0 \quad \text{in } \mathbb{R}^d \times [0,T]. \tag{1.4}$$

Matrix $A$ takes $\mathbb{R}^d$ onto the space

$$\mathcal{L} := \Im(A) \subset \mathbb{R}^d \tag{1.5}$$

of dimension $k := \operatorname{rank} A$. If $k < d$, then Eq. (1.1a) is ultra-parabolic. Ultra-parabolic equations arise in fluid dynamics, combustion theory, and financial mathematics [7]. They describe, in particular, non-stationary transport of matter or temperature in cases when effects of diffusion in some spatial directions are negligible as compared to convection [6]. The pioneering works on equations of the type (1.1a) were done by L. Graetz (1885) and W. Nusselt (1910) who studied the problems of determining the thermal distribution in the laminar flow of an incompressible fluid within cylindrical tubes for the case with both dissipation due to viscosity and horizontal curvature of thermal profiles being neglected [12].

The following notation for the linear spaces of periodic functions is used throughout this work. By $L^p \subset L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and $H^{s,p} \subset H^{s,p}_{\mathrm{loc}}(\mathbb{R}^d)$ we denote the Banach spaces, which consist of 1-periodic functions and are supplemented with the norms $\|u\|_{L^p} = \|u\|_{L^p(\Omega)}$, $\|u\|_{H^{s,p}} = \|u\|_{H^{s,p}(\Omega)}$, where $\Omega$ stands for the unit cube $(0,1)^d$. For $l \geqslant 0$, let $C^l$ be the closed subspace of $u \in C^l(\mathbb{R}^d)$ such that $u$ is 1-periodic with respect to $x_i$, $1 \leqslant i \leqslant d$.

The differential operator $\boldsymbol{A} = \operatorname{div}_x(A\nabla_x \cdot) : C^\infty \mapsto L^2$ is symmetric and non-negative in the Hilbert space $L^2$. By the Friedrichs theorem, it has the self-adjoint extension $\boldsymbol{A} : D(\boldsymbol{A}) \mapsto L^2$. In order to describe the domain of definition $D(\boldsymbol{A})$, we note that $A = O^*\mathcal{D}O$, $\mathcal{D} = \operatorname{diag}\{\lambda_1,\dots,\lambda_k,0,\dots,0\}$, $O^*O = \boldsymbol{I}$, with positive $\lambda_i$. Fix an arbitrary $u \in L^2$ and introduce the function $w \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and the vector field $\partial w \in H^{-1,2}_{\mathrm{loc}}(\mathbb{R}^d)$ defined by

$$w(\boldsymbol{x}) = u(O\boldsymbol{x}), \qquad \partial w = \{\partial_{x_1} w,\dots,\partial_{x_k} w,0,\dots,0\}^\top.$$

A function $u \in L^2$ belongs to $D(\boldsymbol{A})$ if and only if $w \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $\partial w \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Being supplemented with the norm

$$\|u\|^2_{\mathfrak{H}} := \|u\|^2_{L^2} + \left\|A^{1/2}\nabla_x u\right\|^2_{L^2}, \qquad A^{1/2}\nabla_x u(\boldsymbol{x}) := O\mathcal{D}^{1/2}\partial w(O\boldsymbol{x}),$$

$D(\boldsymbol{A})$ becomes the Hilbert space, which will be denoted by $\mathfrak{H}$.

We are now in a position to define an entropy solution of problem (1.1).

**Definition 1.** A function $u \in L^\infty \cap L^2(0,T;\mathfrak{H})$ is an entropy solution of problem (1.1) if and only if the integral inequality

$$\int_Q \left\{\varphi(u)\partial_t\zeta + \psi(u)\boldsymbol{v}\cdot\nabla_x\zeta + \omega(u)\,\mathrm{div}_x(A\nabla_x\zeta) - \varphi''(u)b'(u)\left|A^{1/2}\nabla_x u\right|^2\zeta\right\} d\boldsymbol{x}\,dt$$
$$+ \int_\Omega \varphi(u_0)\zeta(\boldsymbol{x},0)\,d\boldsymbol{x} \geqslant 0 \tag{1.6}$$

holds for all functions $\varphi$, $\psi$, and $\omega$ such that

$$\varphi \in C^2_{\mathrm{loc}}(\mathbb{R}), \qquad \varphi''(u) \geqslant 0, \qquad \psi'(u) = a'(u)\varphi'(u),$$
$$\omega'(u) = b'(u)\varphi'(u), \tag{1.7}$$

and for all non-negative 1-periodic in $\boldsymbol{x}$ test functions $\zeta \in C^2_{\mathrm{loc}}(\mathbb{R}^d \times [0,T])$ such that $\zeta|_{t=T} = 0$.

Along with problem (1.1) we consider its parabolic approximation

$$\mathbb{R}^d \times (0,T): \quad \partial_t u_\varepsilon + \mathrm{div}_x\big(\boldsymbol{v}_\varepsilon a_\varepsilon(u_\varepsilon)\big) - \mathrm{div}_x\big(A\nabla_x b(u_\varepsilon)\big) = \varepsilon\Delta_x u_\varepsilon, \tag{1.8}$$

endowed with the boundary data (1.1b) and (1.1c), where divergence free vector fields $\boldsymbol{v}_\varepsilon \in C^\infty(0,T;C^\infty)$ and smooth functions $a_\varepsilon \in C^\infty(\mathbb{R})$, $\varepsilon > 0$, satisfy the relations

$$\|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{L^1(0,T;H^{1,1})} + \|a_\varepsilon - a\|_{H^{1,1}(0,1)} \to 0, \quad \text{as } \varepsilon \searrow 0. \tag{1.9}$$

It follows from the general theory of second order parabolic equations (see [5]) that this problem has a unique smooth solution. Maximum principle and energy estimate imply the inequalities

$$0 \leqslant u_\varepsilon \leqslant 1 \quad \text{and} \quad \|u_\varepsilon\|_{L^2(0,T;\mathfrak{H})} \leqslant c, \tag{1.10}$$

in which the constant $c$ does not depend on $\varepsilon$.

We aim to prove that problem (1.1) has a unique entropy solution $u$ and that solutions $u_\varepsilon$ of problem (1.8), (1.1b), (1.1c) converge in measure to $u$, as $\varepsilon \searrow 0$. The proof relies on the method of kinetic equation, which allows to reduce quasilinear equations and systems to linear scalar equations on 'distribution' functions involving additional 'kinetic' variables. This method has been created and applied recently to study a wide range of problems, for example, to study the equations of isentropic gas dynamics and $p$-systems [8,10], and the first and second order quasilinear conservation laws [1,3,9,13].

In the present work, we introduce the kinetic formulation in the form that works both for entropy and measure valued solutions of problem (1.1). This formulation is motivated by the notion and properties of the Young measures associated with the sequence $u_\varepsilon$, and its appearance is considered in details in Sections 2 and 3. Before stating it let us recall some facts from the measure theory. Further, $\mathbb{M}(\mathbb{R}^n)$ denotes the Banach space of bounded Radon measures on $\mathbb{R}^n$. Recall that the mapping $\sigma : \mathbb{R}^d_x \times (0,T) \mapsto \mathbb{M}(\mathbb{R}^n)$ is said to be bounded weakly* measurable and 1-periodic if for all $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d_x \times (0,T); C_0(\mathbb{R}^n))$ the function

$$(\boldsymbol{x}, t) \mapsto \int\limits_{\mathbb{R}^n_p} F(\boldsymbol{x}, t, p)\, d\sigma_{x,t}(p)$$

is measurable and

$$\int\limits_{\mathbb{R}^n_p} F(\boldsymbol{x}, t, p)\, d\sigma_{x+e_i,t}(p) = \int\limits_{\mathbb{R}^n_p} F(\boldsymbol{x} - \boldsymbol{e}_i, t, p)\, d\sigma_{x,t}(p)$$

for $i = 1, \dots, d$. Here, we use the standard notation $\sigma_{x,t} = \sigma(\boldsymbol{x}, t)$ as if measures $\sigma_{x,t}$ were parametrized by $(\boldsymbol{x}, t)$, and, in line with the notation from [11], we say that $\sigma \in L^\infty_w(\mathbb{R}^d_x \times (0,T); \mathbb{M}(\mathbb{R}^n))$.

**Problem K** (*Kinetic formulation of problem* (1.1)). Let $f_0 : \mathbb{R}^d_x \times \mathbb{R}_\lambda \mapsto [0,1]$ be a measurable function such that $f_0$ is 1-periodic in $x$, monotone and right continuous with respect to $\lambda$ and

$$f_0(\boldsymbol{x}, \lambda) = 0 \quad \text{for } \lambda < 0 \qquad \text{and} \qquad f_0(\boldsymbol{x}, \lambda) = 1 \quad \text{for } \lambda \geqslant 1. \tag{1.11a}$$

It is necessary to find a distribution function $f \in L^\infty(\mathbb{R}^d_x \times (0,T) \times \mathbb{R}_\lambda)$, a parametrized non-negative measure $\sigma \in L^\infty_w(\mathbb{R}^d_x \times (0,T); \mathbb{M}(\mathbb{R}_\lambda \times \mathcal{L}_q))$, and a non-negative defect measure $M \in \mathbb{M}(\mathbb{R}^d_x \times (0,T) \times \mathbb{R}_\lambda)$ satisfying the following conditions:

(a) Function $f(\boldsymbol{x}, t, \lambda)$ is 1-periodic in $\boldsymbol{x}$, monotone and right continuous in $\lambda \in \mathbb{R}$. Moreover,

$$f(\boldsymbol{x}, t, \lambda) = 0 \quad \text{for } \lambda < 0 \qquad \text{and} \qquad f(\boldsymbol{x}, t, \lambda) = 1 \quad \text{for } \lambda \geqslant 1. \tag{1.11b}$$

In particular, $0 \leqslant f \leqslant 1$ a.e. in $Q \times \mathbb{R}_\lambda$. This means that the Stieltjes measure $\mu_{x,t} = d_\lambda f(\boldsymbol{x}, t, \lambda)$ is a probability measure on $\mathbb{R}_\lambda$, and $\operatorname{spt} \mu_{x,t} \subset [0,1]$.

(b) Parametrized measure $\sigma_{x,t}$ is weakly* measurable and 1-periodic in $\boldsymbol{x}$. It is supported on $[0,1] \times \mathcal{L}_q$ and satisfies the conditions

$$\int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} d\sigma_{x,t}(\lambda, \boldsymbol{q}) = 1, \qquad \int\limits_{Q} \left\{ \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} |\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \right\} d\boldsymbol{x}\, dt < \infty. \tag{1.11c}$$

In particular, the function

$$\chi(\boldsymbol{x}, t, s) := \int\limits_{(-\infty, s] \times \mathcal{L}_q} |\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \tag{1.11d}$$

is 1-periodic in $\boldsymbol{x}$, monotone and right continuous in $s$, and the Stieltjes measure $d_\lambda \chi(\boldsymbol{x}, t, \lambda)$ is supported on $[0,1]$ for a.e. $(\boldsymbol{x}, t) \in \mathbb{R}^d_x \times (0,T)$.

(c) Whenever $g \in C^1_{\mathrm{loc}}(\mathbb{R}_\lambda)$, the function $G:(\boldsymbol{x},t) \mapsto \int_{\mathbb{R}_\lambda} g(\lambda)\,d_\lambda f(\boldsymbol{x},t,\lambda)$ belongs to the Hilbert space $L^2(0,T;\mathfrak{H})$ and the equality

$$A^{1/2}\nabla_x G(\boldsymbol{x},t) = \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} g'(\lambda)\boldsymbol{q}\,d\sigma_{x,t}(\lambda,q) \tag{1.11e}$$

holds for a.e. $(\boldsymbol{x},t) \in \mathbb{R}^d_x \times (0,T)$.

(d) Measure $M \in \mathbb{M}(\mathbb{R}^d_x \times (0,T) \times \mathbb{R}_\lambda)$ is non-negative and 1-periodic in $\boldsymbol{x}$.

(e) Distribution function $f:\mathbb{R}^d_x \times (0,T) \times \mathbb{R}_\lambda \mapsto [0,1]$ satisfies the equations and initial conditions

$$Q \times \mathbb{R}_\lambda: \quad \partial_t f + \mathrm{div}_x\big(a'(\lambda) f \boldsymbol{v} - b'(\lambda) A \nabla_x f\big) + \partial_\lambda\big(b'(\lambda)\partial_\lambda \chi + M\big) = 0, \tag{1.11f}$$

$$\Omega \times \mathbb{R}_\lambda: \quad f(\boldsymbol{x},0,\lambda) = f_0(\boldsymbol{x},\lambda). \tag{1.11g}$$

Equations (1.11f) and (1.11g) are understood in the sense of distributions and can be equivalently collected into the integral formulation

$$\begin{aligned} &\int\limits_{Q\times\mathbb{R}_\lambda} \big\{\partial_t\zeta + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\zeta + b'(\lambda)\,\mathrm{div}_x(A\nabla_x\zeta)\big\} f(\boldsymbol{x},t,\lambda)\,d\boldsymbol{x}\,dt\,d\lambda \\ &\quad + \int\limits_{Q\times\mathbb{R}_\lambda} \partial_\lambda\zeta\,dM + \int\limits_{Q\times\mathbb{R}_\lambda} b'(\lambda)\partial_\lambda\zeta\,d_\lambda\chi(\boldsymbol{x},t,\lambda)\,d\boldsymbol{x}\,dt \\ &\quad + \int\limits_{\Omega\times\mathbb{R}_\lambda} \zeta(\boldsymbol{x},0,\lambda) f_0(\boldsymbol{x},\lambda)\,d\boldsymbol{x}\,d\lambda = 0 \end{aligned} \tag{1.11h}$$

for all 1-periodic in $\boldsymbol{x}$ smooth test functions $\zeta(\boldsymbol{x},t,\lambda)$ vanishing in some neighborhood of the plane $\{t=T\}$ and for sufficiently large $|\lambda|$.

**Remark 2.** It is easy to see that the set of solutions to Problem K is convex.

**Remark 3.** If $u$ is the entropy solution of problem (1.1) with the initial data $u_0$, then it is easy to see that there exists a solution of Problem K with the initial data

$$f_0(\boldsymbol{x},\lambda) = 0 \quad \text{for } \lambda < u_0(\boldsymbol{x}) \qquad \text{and} \qquad f_0(\boldsymbol{x},\lambda) = 1 \quad \text{otherwise} \tag{1.12}$$

such that $f(\boldsymbol{x},t,\lambda) = 0$ for $\lambda < u(\boldsymbol{x},t)$ and $f(\boldsymbol{x},t,\lambda) = 1$ otherwise. Vice versa, if $(f,\sigma,M)$ is the solution of Problem K with the initial data (1.12) and $f$ attains only values 0 and 1, then $u(\boldsymbol{x},t) = \sup\{\lambda\colon\ f(\boldsymbol{x},t,\lambda) = 0\}$ is the entropy solution to problem (1.1) with the initial data $u_0$.

The main result of this paper is the following theorem on existence and uniqueness of solutions of problem (1.1).

**Theorem 4.** *Whenever* $u_0 \in L^\infty$, *problem* (1.1) *has a unique entropy solution* $u \in L^\infty(0,T;L^\infty) \cap L^2(0,T;\mathfrak{H})$.

The proof relies on the following assertions on solvability and uniqueness of solutions of Problem K. The first of them is proved in Section 3. It guarantees the existence of a solution to Problem K provided with the periodic in $\boldsymbol{x}$ initial data $f_0 : \mathbb{R}^d_x \times \mathbb{R}_\lambda \mapsto \{0,1\}$:

**Theorem 5.** *Suppose that the initial distribution* $f_0 : \mathbb{R}^d_x \times \mathbb{R}_\lambda \mapsto [0,1]$ *is periodic in* $\boldsymbol{x}$, *monotone and right continuous in* $\lambda$, *satisfies* (1.11*a*) *and*

$$f_0(\boldsymbol{x},\lambda)\big(1 - f_0(\boldsymbol{x},\lambda)\big) = 0 \quad \text{a.e. in } \mathbb{R}^d_x \times \mathbb{R}_\lambda. \tag{1.13}$$

*In other words,* $f_0$ *attains the values* 0 *and* 1 *only. Then, Problem K has a solution.*

In Section 4, we justify the renormalization procedure for the kinetic equation (1.11f), which is the crucial point of our study. More precisely, we prove the following theorem.

**Theorem 6.** *For any smooth convex on the interval* $[0,1]$ *function* $\varphi$ *there exists a Borel measure* $H_\varphi \in C(\mathbb{R}_\lambda \times Q)^*$ *supported in the strip* $0 \leqslant \lambda \leqslant 1$ *such that the integral inequality*

$$\int\limits_{\mathbb{R}_\lambda \times Q} \varphi(f)\big\{\partial_t \zeta + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\zeta + b'(\lambda)\,\mathrm{div}_x(A\nabla_x\zeta)\big\}\,d\boldsymbol{x}\,dt\,d\lambda$$

$$+ \int\limits_{\mathbb{R}_\lambda \times \Omega} \varphi(f_0)\zeta(\boldsymbol{x},0,\lambda)\,d\boldsymbol{x}\,d\lambda - \int\limits_{\mathbb{R}_\lambda \times Q} \partial_\lambda\zeta\,dH_\varphi(\boldsymbol{x},t,\lambda) \leqslant 0 \tag{1.14}$$

*holds for any* 1*-periodic in* $\boldsymbol{x}$ *non-negative smooth function* $\zeta(\boldsymbol{x},t,\lambda)$, *which vanishes in a neighborhood of the plane* $t = T$ *and for sufficiently large* $|\lambda|$.

In Section 5, by means of Theorem 6 we obtain the following theorem.

**Theorem 7.** *Under the assumptions of Theorem* 5, *solutions to Problem K satisfy the equality*

$$f(\boldsymbol{x},t,\lambda)\big(1 - f(\boldsymbol{x},t,\lambda)\big) = 0 \quad \text{a.e. in } \mathbb{R}^d_x \times [0,T] \times \mathbb{R}_\lambda. \tag{1.15}$$

*Moreover, if* $(f,\sigma,M)$ *and* $(f',\sigma',M')$ *are the solutions of Problem K with the same initial data* $f_0$, *then* $f = f'$ *a.e. in* $Q \times \mathbb{R}_\lambda$.

It is clear that Theorem 4 is the consequence of Theorems 5 and 7 and Remark 3.

## 2. Preliminaries

In this section, we consider in details the properties of Young measures associated with a sequence of solutions $u_\varepsilon : \mathbb{R}^d_x \times (0,T) \mapsto [0,1]$ of problem (1.8), (1.1b), (1.1c). We start with the observation that, by the Tartar theorem [14], [11, Chapter 3], there exists a subsequence still denoted by $u_\varepsilon$ and a family of probability Radon measures $\mu_{x,t}$ supported uniformly on $[0,1]$ such that

$$g(u_\varepsilon) \to \bar{g} \quad \text{weakly* in } L^\infty(Q), \qquad \bar{g} = \int_{\mathbb{R}_\lambda} g(\lambda)\, d\mu_{x,t}(\lambda) \tag{2.1}$$

for all $g \in C(\mathbb{R}_\lambda)$. The mapping $(\boldsymbol{x}, t) \mapsto \mu_{x,t}$ is weakly* measurable and 1-periodic in $\boldsymbol{x}$.

Set $\boldsymbol{q}_\varepsilon := A^{1/2}\nabla_x u_\varepsilon$. The vector fields $\boldsymbol{q}_\varepsilon : \mathbb{R}^d_x \times (0,T) \mapsto \mathcal{L}$ are measurable and 1-periodic in $\boldsymbol{x}$. From (1.10) it follows that the sequence $(u_\varepsilon, \boldsymbol{q}_\varepsilon)$ is bounded in $L^2$, which along with the Ball theorem [2] yields the following lemma.

**Lemma 8.** *There exists a subsequence still denoted by $(u_\varepsilon, \boldsymbol{q}_\varepsilon)$ and a measure-valued 1-periodic in $\boldsymbol{x}$ function $\sigma \in L^\infty_w(Q, \mathbb{M}(\mathbb{R}_\lambda \times \mathcal{L}_q))$ such that for all continuous functions $g : \mathbb{R}_\lambda \times \mathcal{L}_q \mapsto \mathbb{R}$ satisfying the growth condition $|g(\lambda, \boldsymbol{q})| \leqslant c(1 + |\lambda| + |\boldsymbol{q}|)^p$, $0 \leqslant p < 2$, we have $g(u_\varepsilon, \boldsymbol{q}_\varepsilon) \to \bar{g}$ weakly in $L^r(Q)$, $1 < r \leqslant 2/p$, $\bar{g} = \int_{\mathbb{R}_\lambda \times \mathcal{L}_q} g(\lambda, \boldsymbol{q})\, d\sigma_{x,t}(\lambda, \boldsymbol{q})$ for a.e. $(\boldsymbol{x}, t) \in Q$, and the probability measure $\sigma_{x,t}$ is supported in $[0,1] \times \mathcal{L}_q$.*

**Lemma 9.** *Under the above assumptions there exists a mapping $\nu \in L^1_w(Q; \mathbb{M}(\mathbb{R}_\lambda))$ and a function $\bar{\varphi} \in L^1(Q)$ such that for all $g \in C(\mathbb{R}_\lambda)$, we have*

$$\int_{\mathbb{R}_\lambda} g(\lambda)\, d\nu_{x,t}(\lambda) = \int_{\mathbb{R}_\lambda \times \mathcal{L}_q} g(\lambda) |\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}), \tag{2.2}$$

$$\left| \int_{\mathbb{R}_\lambda} g(\lambda)\, d\nu_{x,t}(\lambda) \right| \leqslant \|g\|_{C(\mathbb{R}_\lambda)} \bar{\varphi}(\boldsymbol{x}, t) \quad \textit{for a.e. } (\boldsymbol{x}, t) \in Q. \tag{2.3}$$

*The mapping $\nu$ is 1-periodic in $\boldsymbol{x}$ and* $\operatorname{spt} \nu_{x,t} \subset [0,1]$ *for a.e. $(\boldsymbol{x}, t) \in Q$.*

In the formulation of the lemma, $L^1_w(Q; \mathbb{M}(\mathbb{R}_\lambda))$ denotes the space of weakly* measurable mappings $\nu : Q \mapsto \mathbb{M}(\mathbb{R}_\lambda)$ such that for any $F \in L^\infty(Q; C_0(\mathbb{R}_\lambda))$ the integral $\int_Q |\int_{\mathbb{R}_\lambda} F\, d\nu_{x,t}(\lambda)|\, d\boldsymbol{x}\, dt$ is finite.

**Proof.** Let a non-negative function $h \in C^\infty_0(\mathbb{R})$ be satisfying the conditions $sh'(s) \geqslant 0$, $h(s) = 1$ when $|s| \leqslant 1$, $h(s) = 0$ when $|s| \geqslant 2$. It is clear that $|\boldsymbol{q}|^2 h(n^{-1}|\boldsymbol{q}|) \nearrow |\boldsymbol{q}|^2$, as $n \nearrow \infty$, and that

$$\int_Q |\boldsymbol{q}_\varepsilon|^2 h\bigl(n^{-1}|\boldsymbol{q}_\varepsilon|\bigr)\, d\boldsymbol{x}\, dt \leqslant \int_Q |\boldsymbol{q}_\varepsilon|^2\, d\boldsymbol{x}\, dt \leqslant C_q < \infty.$$

From this we conclude that non-negative functions $\varphi_n(\boldsymbol{x}, t) = |\boldsymbol{q}_\varepsilon|^2 h(n^{-1}|\boldsymbol{q}_\varepsilon|)$, $n = 1, 2, \ldots,$ satisfy the inequalities

$$\varphi_n \leqslant \varphi_{n+1} \quad \text{and} \quad \int_Q \varphi_n(\boldsymbol{x}, t)\, d\boldsymbol{x}\, dt \leqslant C_q \quad \text{for } n \geqslant 1. \tag{2.4}$$

Since $|\cdot|^2 h(n^{-1}\cdot) \in C(\mathbb{R}_\lambda \times \mathcal{L}_q)$, we can assume that $\varphi_n \to \overline{\varphi_n}$ weakly* in $L^\infty(Q)$, as $\varepsilon \searrow 0$, where

$$\overline{\varphi_n}(\boldsymbol{x}, t) = \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} |\boldsymbol{q}|^2 h\big(n^{-1}|\boldsymbol{q}|\big)\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \quad \text{for a.e. } (\boldsymbol{x}, t) \in Q. \tag{2.5}$$

Inequalities (2.4) imply $\overline{\varphi_n} \leqslant \overline{\varphi_{n+1}}$ and $\|\overline{\varphi_n}\|_{L^1(Q)} \leqslant C_q$. By the Fatou theorem, there exists $\bar{\varphi} \in L^1(Q)$ such that $\overline{\varphi_n}(\boldsymbol{x}, t) \nearrow \bar{\varphi}(\boldsymbol{x}, t)$ a.e. in $Q$, which along with (2.5) yields

$$\int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} |\boldsymbol{q}|^2 h\big(n^{-1}|\boldsymbol{q}|\big)\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \nearrow \bar{\varphi}(\boldsymbol{x}, t)$$

for a.e. $(\boldsymbol{x}, t) \in Q$. Since $|\boldsymbol{q}|^2 h(n^{-1}|\boldsymbol{q}|) \nearrow |\boldsymbol{q}|^2$, as $n \nearrow \infty$, the Fatou theorem yields

$$\int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} |\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) = \bar{\varphi}(\boldsymbol{x}, t) < \infty \quad \text{a.e. } (\boldsymbol{x}, t) \in Q. \tag{2.6}$$

Next, note that for all $g \in C(\mathbb{R}_\lambda)$, we have $|g(\lambda)||\boldsymbol{q}|^2 h(n^{-1}|\boldsymbol{q}|) \leqslant \|g\|_{C(\mathbb{R}_\lambda)} |\boldsymbol{q}|^2$ and $g(\lambda)|\boldsymbol{q}|^2 h(n^{-1}|\boldsymbol{q}|) \to g(\lambda)|\boldsymbol{q}|^2$, as $n \nearrow \infty$. From this, (2.6), and the Lebesgue dominated convergence theorem we conclude that

$$\int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} g(\lambda)|\boldsymbol{q}|^2 h\big(n^{-1}|\boldsymbol{q}|\big)\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \to \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} g(\lambda)|\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}), \tag{2.7}$$

as $n \nearrow \infty$, and that

$$\Bigg| \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} g(\lambda)|\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \Bigg| \leqslant \|g\|_{C(\mathbb{R}_\lambda)} \overline{\varphi(\boldsymbol{x}, t)}. \tag{2.8}$$

Therefore, the function

$$\Phi_g : (\boldsymbol{x}, t) \mapsto \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} g(\lambda)|\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \tag{2.9}$$

belongs to $L^1(Q)$ and satisfies the inequalities

$$\big|\Phi_g(\boldsymbol{x}, t)\big| \leqslant \|g\|_{C(\mathbb{R}_\lambda)} \overline{\varphi(\boldsymbol{x}, t)} \quad \text{a.e. in } \mathbb{R}^d_x \times (0, T). \tag{2.10}$$

Let $E \subset \mathbb{R}^d_x \times (0, T)$ be a measurable set with a complement of zero measure such that $\bar{\varphi}(\boldsymbol{x}, t) < \infty$ for each $(\boldsymbol{x}, t) \in E$. Whenever $(\boldsymbol{x}, t) \in E$, the mapping $g \mapsto \Phi_g(\boldsymbol{x}, t)$ is linear and continuous on $C(\mathbb{R}_\lambda)$. By the Riesz theorem, there exists a Radon measure $\nu_{x,t} \in \mathbb{M}(\mathbb{R}_\lambda)$ such that the identity

$$\int\limits_{\mathbb{R}_\lambda} g(\lambda)\, d\nu_{x,t}(\lambda) = \Phi_g(\boldsymbol{x}, t) \tag{2.11}$$

holds for all compactly supported $g \in C(\mathbb{R}_\lambda)$ and $(x, t) \in E$. Note that (2.2) and (2.3) follow from (2.9) and (2.10). It remains to prove that $\operatorname{spt} \nu_{x,t} \subset [0, 1]$. Choose an arbitrary open interval $I \supset [0,1]$ and a non-negative function $\eta \in C(\mathbb{R})$ such that $\eta(s) = 0$ when $s \in [0,1]$, and $\eta(s) = 1$ when $s \in \mathbb{R} \setminus I$. Since $0 \leqslant u_\varepsilon \leqslant 1$, we have the identity $\eta(u_\varepsilon) \equiv 0$, which along with Lemma 8 yields

$$w\text{-}\lim_{\varepsilon \searrow 0} \eta(u_\varepsilon)|\boldsymbol{q}_\varepsilon|^2 h\big(n^{-1}|\boldsymbol{q}_\varepsilon|\big) = \int_{\mathbb{R}_\lambda \times \mathcal{L}_q} \eta(\lambda)|\boldsymbol{q}|^2 h\big(n^{-1}|\boldsymbol{q}|\big)\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) = 0.$$

From this, (2.7), and (2.8) we conclude that

$$\int_{(\mathbb{R}_\lambda \setminus I) \times \mathcal{L}_q} |\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \leqslant \int_{\mathbb{R}_\lambda \times \mathcal{L}_q} \eta(\lambda)|\boldsymbol{q}|^2\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) = 0$$

for all $(\boldsymbol{x}, t) \in E$ and the lemma follows. □

**Lemma 10.** *There exist subsequence $(u_\varepsilon, \boldsymbol{q}_\varepsilon)$ and non-negative Radon measures $M_0$ and $M$ on $\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda$ such that $M_0$ and $M$ are 1-periodic in $\boldsymbol{x}$ and supported in $\mathbb{R}^d_x \times [0,T] \times [0,1]$, and the equalities*

$$\lim_{\varepsilon \searrow 0} \int_Q g(\boldsymbol{x}, t, u_\varepsilon)\big(|\boldsymbol{q}_\varepsilon|^2 + \varepsilon|\nabla_x u_\varepsilon|^2\big)\, d\boldsymbol{x}\, dt = \int_{Q \times \mathbb{R}_\lambda} g(\boldsymbol{x}, t, \lambda)\, dM_0, \tag{2.12}$$

$$\int_{Q \times \mathbb{R}_\lambda} g\, dM_0 = \int_Q \left\{ \int_{\mathbb{R}_\lambda} g\, d\nu_{x,t}(\lambda) \right\} d\boldsymbol{x}\, dt + \int_{Q \times \mathbb{R}_\lambda} g\, dM \tag{2.13}$$

*hold for any 1-periodic in $\boldsymbol{x}$ function $g \in C(\mathbb{R}^d_x \times (0,T) \times \mathbb{R}_\lambda)$.*

**Proof.** Let us consider the functional $M_\varepsilon$ defined by

$$\langle M_\varepsilon, g \rangle = \int_{\mathbb{R}^d_x \times (0,T)} g(\boldsymbol{x}, t, u_\varepsilon)\big(|\boldsymbol{q}_\varepsilon|^2 + \varepsilon|\nabla_x u_\varepsilon|^2\big)\, d\boldsymbol{x}\, dt. \tag{2.14}$$

It follows from (1.10) that

$$\big|\langle M_\varepsilon, g \rangle\big| \leqslant c\, \mathrm{diam}(K) \|g\|_{C(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda)} \tag{2.15}$$

for every function $g \in C(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda)$ supported in some compact $K \subset \mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda$. Moreover, $\langle M_\varepsilon, g \rangle = 0$ for each continuous function $g$, which vanishes on $\mathbb{R}^d_x \times [0,T] \times [0,1]$. By the Riesz theorem, $M_\varepsilon$ is a Radon measure in $\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda$ supported in $\mathbb{R}^d_x \times [0,T] \times [0,1]$. Clearly, it is 1-periodic in $\boldsymbol{x}$. After passing to a subsequence, we can assume that the sequence $M_\varepsilon$ converges weakly* to a Radon measure $M_0$ in $\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda$, as $\varepsilon \searrow 0$. It is clear that the measure $M_0$ is 1-periodic in $\boldsymbol{x}$ and that $\operatorname{spt} M_0 \subset \mathbb{R}^d_x \times [0,T] \times [0,1]$. Next, note that the inequality

$$\int_{\mathbb{R}^d_x \times (0,T)} g_0(\boldsymbol{x}, t) g_1(u_\varepsilon)\big(|\boldsymbol{q}_\varepsilon|^2 + \varepsilon|\nabla u_\varepsilon|^2\big)\, d\boldsymbol{x}\, dt$$

$$\geqslant \int_{\mathbb{R}^d_x \times (0,T)} g_0(\boldsymbol{x}, t) g_1(u_\varepsilon)|\boldsymbol{q}_\varepsilon|^2 h\big(n^{-1}|\boldsymbol{q}_\varepsilon|\big)\, d\boldsymbol{x}\, dt$$

holds for all non-negative compactly supported functions $g_0 \in C(\mathbb{R}^d_x \times \mathbb{R}_t)$, $g_1 \in C(\mathbb{R}_\lambda)$ and integer $n \geqslant 1$. Passing to the limit in both the sides of this inequality, as $\varepsilon \searrow 0$, along a suitable subsequence, we arrive at

$$\langle M_0, g_0 g_1\rangle \geqslant \int\limits_{\mathbb{R}^d_x \times (0,T)} g_0(\boldsymbol{x}, t)\left\{\int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} g_1(\lambda)|\boldsymbol{q}|^2 h\big(n^{-1}|\boldsymbol{q}|\big)\, d\sigma_{x,t}(\lambda, \boldsymbol{q})\right\} d\boldsymbol{x}\, dt.$$

From this, (2.7), (2.11), and (2.14) we conclude that the inequality

$$\langle M_0, g_0 g_1\rangle \geqslant \int\limits_{\mathbb{R}^d_x \times (0,T)} g_0(\boldsymbol{x}, t)\left\{\int\limits_{\mathbb{R}_\lambda} g_1(\lambda)\, d\nu_{x,t}(\lambda)\right\} d\boldsymbol{x}\, dt \tag{2.16}$$

holds for all non-negative compactly supported functions $g_0 \in C(\mathbb{R}^d_x \times \mathbb{R}_t)$, $g_1 \in C(\mathbb{R}_\lambda)$. On the other hand, the formula

$$\big\langle M^*, g\big\rangle = \int\limits_{\mathbb{R}^d_x \times (0,T)} \left\{\int\limits_{\mathbb{R}_\lambda} g(\boldsymbol{x}, t, \lambda)\, d\nu_{x,t}(\lambda)\right\} d\boldsymbol{x}\, dt \quad \forall g \in C_c\big(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda\big)$$

defines a non-negative Radon measure on $\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda$ with $\operatorname{spt} M^* \subset \mathbb{R}^d_x \times [0, T] \times [0, 1]$. It follows from this and (2.16) that the defect measure $M = M_0 - M^*$ satisfies the inequality $\langle M, g_0 g_1\rangle \equiv \langle M_0, g_0 g_1\rangle - \langle M^*, g_0 g_1\rangle \geqslant 0$ for all non-negative functions $g_0 g_1$ with $g_0 \in C_c(\mathbb{R}^d_x \times \mathbb{R}_t)$ and $g_1 \in C_c(\mathbb{R}_\lambda)$. Note that the linear span of the set of such functions is dense in $C_c(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda)$ and, consequently, the inequality $\langle M, g\rangle \geqslant 0$ holds for all non-negative functions $g \in C_c(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}_\lambda)$. Hence $M \geqslant 0$, and the lemma follows. □

The next lemma shows that the measure $M_0$ does not concentrate near the plane $\{t = 0\}$.

**Lemma 11.** *Measure $M_0$ defined in Lemma* 10 *satisfies the limiting relation*

$$\lim_{\tau \searrow 0} \int\limits_{\Omega \times [0,\tau] \times \mathbb{R}_\lambda} dM_0(\boldsymbol{x}, t, \lambda) = 0.$$

**Proof.** We start with the observation that the functions $u_\varepsilon(\cdot\,, t)$, $t \in [0, T]$, $\varepsilon > 0$, are equicontinuous in the weak topology. Multiplying both the sides of Eq. (1.8) by a function $\zeta \in C^\infty$ and integrating over the cylinder $\Omega \times [0, t]$, we arrive at

$$\begin{aligned}\int\limits_\Omega \partial_t u_\varepsilon(\boldsymbol{x}, t)\zeta(\boldsymbol{x})\, d\boldsymbol{x} &= \int\limits_\Omega \big(\boldsymbol{v}_\varepsilon(\boldsymbol{x}, t) a_\varepsilon\big(u_\varepsilon(\boldsymbol{x}, t)\big)\big) \cdot \nabla_x \zeta(\boldsymbol{x})\, d\boldsymbol{x} \\ &\quad + \int\limits_\Omega \Big(b\big(u_\varepsilon(\boldsymbol{x}, t)\big) \operatorname{div}_x\big(A \nabla_x \zeta(\boldsymbol{x})\big) \\ &\quad + \varepsilon u_\varepsilon(\boldsymbol{x}, t) \Delta_x \zeta(\boldsymbol{x})\Big)\, d\boldsymbol{x} \quad \text{for } t \in (0, T).\end{aligned} \tag{2.17}$$

Since $\|\boldsymbol{v}_\varepsilon(t)\|_{H^{1,1}} \to \|\boldsymbol{v}(t)\|_{H^{1,1}}$ in $L^1(0, T)$, $a_\varepsilon \to a$ uniformly on every interval and $0 \leqslant u_\varepsilon \leqslant 1$, there exist a function $\rho \in L^1(0, T)$ and constants $c_a$ and $c_b$ such that

$\|\boldsymbol{v}_\varepsilon(\cdot\,,t)\|_{L^1(0,T)} \leqslant \rho(t)$, $|a_\varepsilon(u_\varepsilon)| \leqslant c_a$, and $|b(u_\varepsilon)| \leqslant c_b$. From this and (2.17) we conclude that

$$\left|\int_\Omega \partial_t u_\varepsilon(\boldsymbol{x},t)\zeta(\boldsymbol{x})\,d\boldsymbol{x}\right| \leqslant \|\zeta\|_{C^2}\big(c_a\rho(t) + c_b\|A\| + \varepsilon\big).$$

Since the embedding $H^{s,2} \hookrightarrow C^2$ is compact for $s > s_d = [d/2] + 5/2$, we obtain

$$\big\|\partial_t u_\varepsilon(\cdot\,,t)\big\|_{H^{-s,2}} \leqslant c(\rho(t)+1) \quad \text{for } t \in (0,T) \text{ and } s > s_d. \tag{2.18}$$

Hence, the mappings $u_\varepsilon : [0,T] \mapsto H^{-s,2}$ are equicontinuous. In particular, $u_\varepsilon(t) \to u_0$ in $H^{-s,2}$ uniformly with respect to $\varepsilon \in (0,1)$, as $t \searrow 0$. On the other hand, the values of functions $u_\varepsilon(\cdot,t)$ belong to the interval $\|u_\varepsilon\|_{L^\infty} \leqslant 1$, which is a compact subset of $H^{-s,2}$. By the Arcel theorem, the set $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively compact in $C(0,T;H^{-s,2})$. Hence, there exists a subsequence still denoted by $u_\varepsilon$ and a function $u^* \in L^\infty$ such that $u_\varepsilon(t) \to u^*(t)$ in $H^{-s,1}$ uniformly on the segment $[0,T]$. Moreover, $u^*(t) \to u_0$ in $H^{-s,2}$, as $t \searrow 0$. Hence,

$$u_\varepsilon(\cdot,t) \to u^*(t) \quad \text{as } \varepsilon \searrow 0 \qquad \text{and} \qquad u^*(t) \to u_0 \quad \text{as } t \searrow 0,$$
$$\text{weakly in } L^2. \tag{2.19}$$

Fix an arbitrary vector $z \in \mathbb{R}^d$. Multiplying both sides of Eq. (1.8) by $u_\varepsilon$ and integrating over $\Omega \times (0,T)$, we obtain

$$\frac{1}{2}\big\|u_\varepsilon(\cdot\,,t)\big\|^2_{L^2} + \int\limits_{(\Omega+z)\times(0,t)} \big(b'(u_\varepsilon)|q_\varepsilon|^2 + \varepsilon|\nabla_x u_\varepsilon|^2\big)\,d\boldsymbol{x}\,dt = \frac{1}{2}\big\|u_\varepsilon(\cdot\,,t)\big\|^2_{L^2}.$$

Using (2.14) and noting that the measure $M_\varepsilon$ is supported in $\mathbb{R}^d_x \times [0,T] \times [0,1]$, we can rewrite this equality in the form

$$\frac{1}{2}\big\|u_\varepsilon(\cdot\,,t)\big\|^2_{L^2} + \int\limits_{(\Omega+z)\times(-\delta,t)\times\mathbb{R}_\lambda} dM_\varepsilon(\boldsymbol{x},t,\lambda) = \frac{1}{2}\|u_0\|^2_{L^2} \tag{2.20}$$

with an arbitrary positive $\delta$. Since $M_\varepsilon$ converges weakly to the measure $M_0$ and the sequence $u_\varepsilon(\cdot\,,t)$ converges weakly to $u^*(\cdot\,,t)$, we have

$$\int\limits_{(\Omega+z)\times(-\delta,t)\times\mathbb{R}_\lambda} dM_0(\boldsymbol{x},t,\lambda) \leqslant \limsup_{\varepsilon\searrow 0} \int\limits_{(\Omega+z)\times(-\delta,t)\times\mathbb{R}_\lambda} dM_\varepsilon(\boldsymbol{x},t,\lambda).$$

On the other hand, relations (2.19) imply the inequality

$$\big\|u^*(\cdot\,,t)\big\|_{L^2} \leqslant \liminf_{\varepsilon\searrow 0}\big\|u_\varepsilon(\cdot\,,t)\big\|_{L^2},$$

which along with (2.20) gives

$$\int\limits_{(\Omega+z)\times(-\delta,t)\times\mathbb{R}_\lambda} dM_0(\boldsymbol{x},t,\lambda) \leqslant \frac{1}{2}\|u_0\|^2_{L^2} - \frac{1}{2}\big\|u^*(\cdot\,,t)\big\|^2_{L^2} \quad \text{for every } z \in \mathbb{R}^d.$$

It remains to note that

$$\liminf_{t\searrow 0}\left\|u^*(\cdot\,,t)\right\|_{L^2}\geqslant\|u_0\|_{L^2},$$

and the lemma follows. □

Let us introduce the distribution function $f$ of the Young measure $\mu_{x,t}$,

$$f(\boldsymbol{x},t,\lambda)=\int_{\mathbb{R}_s}1_{s\leqslant\lambda}\,d\mu_{x,t}(s). \tag{2.21}$$

We observe that the distribution function $f$ satisfies all conditions in item (a) of the formulation of Problem K. The next lemma establishes the relation between the function $f$ and the Young measure $\sigma_{x,t}$.

**Lemma 12.** *The identity*

$$A^{1/2}\nabla_x f(\boldsymbol{x},t,\lambda)=-\int_{\mathcal{L}_q}\boldsymbol{q}\,d\sigma_{x,t}(\lambda,\boldsymbol{q})$$

*holds true in the sense of distributions.*

**Proof.** Consider $\varphi(u_\varepsilon)\to\varphi^*$ weakly* in $L^\infty(Q)$, $A^{1/2}\nabla_x\varphi(u_\varepsilon)\to G^*$ weakly in $L^2(Q)$, as $\varepsilon\searrow 0$, where $\varphi$ is an arbitrary smooth function. For an arbitrary smooth 1-periodic in $\boldsymbol{x}$ vector-function $\boldsymbol{\zeta}$, one has both

$$\int_Q\boldsymbol{\zeta}\cdot A^{1/2}\nabla_x\varphi(u_\varepsilon)\,d\boldsymbol{x}\,dt\underset{\varepsilon\searrow 0}{\longrightarrow}\int_{Q\times\mathbb{R}_\lambda}\operatorname{div}_x\bigl(A^{1/2}\boldsymbol{\zeta}\bigr)\varphi'(\lambda)f(\boldsymbol{x},t,\lambda)\,d\boldsymbol{x}\,dt\,d\lambda$$

and

$$\int_Q\boldsymbol{\zeta}\cdot A^{1/2}\nabla_x\varphi(u_\varepsilon)\,d\boldsymbol{x}\,dt\underset{\varepsilon\searrow 0}{\longrightarrow}\int_{Q\times\mathbb{R}_\lambda\times\mathcal{L}_q}\boldsymbol{\zeta}\varphi'(\lambda)\cdot\boldsymbol{q}\,d\sigma_{x,t}(\lambda,\boldsymbol{q})\,d\boldsymbol{x}\,dt,$$

which completes the proof. □

## 3. Proof of Theorem 5

Choose an arbitrary smooth function $\varphi\in C_0^\infty(\mathbb{R})$. Let

$$\Phi(\lambda)=-\int_\lambda^{+\infty}\varphi\,ds,\qquad\Psi_\varepsilon(\lambda)=-\int_\lambda^{+\infty}a_\varepsilon'\varphi\,ds,\qquad w(\lambda)=-\int_\lambda^{+\infty}b'\varphi\,ds. \tag{3.1}$$

Multiplying both sides of Eq. (1.8) by $\varphi(u_\varepsilon)\eta(\boldsymbol{x},t)$, where $\eta\in C^\infty(Q)$, $\eta(\boldsymbol{x}+\boldsymbol{e}_i,t)=\eta(\boldsymbol{x},t)$, and $\eta(\boldsymbol{x},T)=0$, and integrating over $Q$, we obtain

$$\int_Q \big\{\Phi(u_\varepsilon)\partial_t\eta + \Psi_\varepsilon(u_\varepsilon)\boldsymbol{v}_\varepsilon \cdot \nabla_x\eta + w(u_\varepsilon)\,\mathrm{div}_x(A\nabla_x\eta)$$
$$+\,\varepsilon\Phi(u_\varepsilon)\Delta_x\eta - \Phi''(u_\varepsilon)b'(u_\varepsilon)\big|A^{1/2}\nabla_x u_\varepsilon\big|^2\eta - \varepsilon\Phi''(u_\varepsilon)|\nabla_x u_\varepsilon|^2\eta\big\}\,d\boldsymbol{x}\,dt$$
$$+\int_\Omega \Phi(u_{0\varepsilon})\eta(\boldsymbol{x},0)\,d\boldsymbol{x} = 0.$$

As $\varepsilon \searrow 0$, on the strength of Lemmas 8–10, we derive

$$\int_{Q\times\mathbb{R}_\lambda} \big\{\Phi(\lambda)\partial_t\eta + \Psi(\lambda)\boldsymbol{v}\cdot\nabla_x\eta + w(\lambda)\,\mathrm{div}_x(A\nabla_x\eta)\big\}\,d\mu_{x,t}(\lambda)\,d\boldsymbol{x}\,dt$$
$$-\int_{Q\times\mathbb{R}_\lambda} \eta\Phi''(\lambda)b'(\lambda)\,d\nu_{x,t}(\lambda)\,d\boldsymbol{x}\,dt$$
$$-\int_{Q\times\mathbb{R}_\lambda} \Phi''(\lambda)\eta\,dM + \int_\Omega \Phi(u_0)\eta(\boldsymbol{x},0)\,d\boldsymbol{x} = 0, \tag{3.2}$$

where $\Psi(\lambda) = -\int_\lambda^{+\infty} a'(s)\varphi(s)\,ds$, and conclude that the parametrized measure $\sigma_{x,t}$ and the defect measure $M$ satisfy conditions of items (b) and (d) of formulation of Problem K. Substituting (3.1) into (3.2), using the notions of the Stieltjes integrals with respect to the measures $d_\lambda f$ and $d_\lambda\chi$ (see items (a) and (b) of the formulation of Problem K) and the equality

$$\int_{\mathbb{R}_\lambda}\left(\int_\lambda^{+\infty}\zeta(s)\,ds\right)d_\lambda f(\boldsymbol{x},t,\lambda) = \int_{\mathbb{R}_\lambda}\zeta(\lambda)f(\boldsymbol{x},t,\lambda)\,d\lambda$$

that holds for a.e. $(\boldsymbol{x},t)\in\mathbb{R}^d\times[0,T]$ for an arbitrary $\zeta\in C_0(\mathbb{R})$ on the strength of the theory of the Stieltjes integral, we arrive at the identity

$$\int_{Q\times\mathbb{R}_\lambda} \big(\partial_t(\varphi\eta) + a'(\lambda)\boldsymbol{v}\cdot\nabla_x(\varphi\eta) + b'(\lambda)\,\mathrm{div}_x\big(A\nabla_x(\varphi\eta)\big)\big)f(\boldsymbol{x},t,\lambda)\,d\boldsymbol{x}\,dt\,d\lambda$$
$$+\int_{\Omega\times\mathbb{R}_\lambda}\varphi\eta_0 f_0\,d\boldsymbol{x}\,d\lambda + \int_{Q\times\mathbb{R}_\lambda}\partial_\lambda(\varphi\eta)\,dM$$
$$+\int_{Q\times\mathbb{R}_\lambda} b'(\lambda)\partial_\lambda(\varphi\eta)\,d_\lambda\chi(\boldsymbol{x},t,\lambda)\,d\boldsymbol{x}\,dt = 0. \tag{3.3}$$

The linear span of $\{\varphi\eta\}$ is dense in $C^\infty(\mathbb{R}_\lambda\times Q)$, therefore, (3.3) is valid with a test function $\zeta(\lambda,x,t)$ on the place of $(\varphi\eta)$. Thus, item (e) of formulation of Problem K is fulfilled.

In order to finish the justification of the theorem, it remains to notice that the condition in item (c) of the formulation of Problem K holds on the strength of Lemma 12.

## 4. Proof of Theorem 6

The proof is divided into five steps.

**Step 1.** *The smoothing of the kinetic equation*. Introduce the mollifier $\omega \in C_0^\infty(\mathbb{R})$, $\|\omega\|_{L^1(\mathbb{R}^1)} = 1$ that is a non-negative smooth function with a compact support on $[0,1]$. For any continuous function $f : \mathbb{R}_x^d \times \mathbb{R}_t^+ \times \mathbb{R}_\lambda \mapsto \mathbb{R}$ we denote

$$f_s(\boldsymbol{x},t,\cdot) = \omega_s * f(\boldsymbol{x},t,\cdot), \qquad f_\tau(\boldsymbol{x},\cdot,\lambda) = \omega_\tau * f(\boldsymbol{x},\cdot,\lambda), \quad \text{and}$$
$$f_h(\cdot,t,\lambda) = \omega_h * \cdots * \omega_h * f(\cdot,t,\lambda).$$

Further we write $f_{\alpha\beta}$ instead of $(f_\alpha)_\beta$ for $\alpha,\beta = h,s,\tau$. Denote by $Q_\tau$ the cylinder $Q_\tau = Q \cap \{\tau < t < T-\tau\}$. Set

$$\zeta(\boldsymbol{x},t,\lambda) = \omega_{sh\tau}(\bar{\boldsymbol{x}} - \boldsymbol{x}, \bar\lambda - \lambda, \bar t - t),$$

where

$$\omega_s(\lambda) = \frac{1}{s}\omega\left(\frac{\lambda}{s}\right), \qquad \omega_h(\boldsymbol{x}) = \frac{1}{h^d}\omega\left(\frac{x_1}{h}\right)\cdots\omega\left(\frac{x_d}{h}\right), \qquad \omega_\tau = \frac{1}{\tau}\omega\left(\frac{t}{\tau}\right),$$

and

$$\omega_{sh\tau} = \omega_s\omega_h\omega_\tau.$$

Further we also write $\omega_{\alpha\beta}$ instead of $\omega_\alpha\omega_\beta$ for $\alpha,\beta = h,s,\tau$.

Substituting $\zeta(\boldsymbol{x},t,\lambda)$ on the place of a test function into (1.11h), we obtain the following equation for the smoothed distribution function, where we write $\lambda$, $\boldsymbol{x}$, and $t$ instead of $\bar\lambda$, $\bar{\boldsymbol{x}}$, and $\bar t$:

$$\begin{aligned}&\partial_t f_{sh\tau} + a'(\lambda)\boldsymbol{v}\cdot\nabla_x f_{sh\tau} - b'(\lambda)\,\mathrm{div}_x(A\nabla_x f_{sh\tau}) + \partial_\lambda\big(b'(\lambda)\partial_\lambda\chi_{sh\tau} + M_{sh\tau}\big)\\ &\quad = R_1^{(sh\tau)} + R_2^{(sh\tau)} + R_3^{(sh\tau)} \quad \text{in } Q_\tau\times\mathbb{R}_\lambda,\end{aligned} \tag{4.1}$$

where the rest terms are given by the formulas

$$\begin{aligned}R_1^{(sh\tau)} &= \mathrm{div}_x\big(a'(\lambda)\boldsymbol{v} f_{sh\tau}\big) - \mathrm{div}_x(a'\boldsymbol{v}f)_{sh\tau},\\ R_2^{(sh\tau)} &= \partial_\lambda\big(b'\partial_\lambda\chi_{sh\tau} - (b'\partial_\lambda\chi_{h\tau})_s\big),\\ R_3^{(sh\tau)} &= -b'(\lambda)\,\mathrm{div}_x(A\nabla_x f_{sh\tau}) + \big(b'\,\mathrm{div}_x(A\nabla_x f_{h\tau})\big)_s.\end{aligned}$$

**Step 2.** *Renormalization of the smoothed kinetic equation*. Let $\varphi \in C^2(\mathbb{R})$ be an arbitrary convex on $[0,1]$ function. Multiplying both sides of (4.1) by $\varphi'(f_{sh\tau})$, we obtain the equation

$$\begin{aligned}&\partial_t\varphi(f_{sh\tau}) + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\varphi(f_{sh\tau}) - b'(\lambda)\,\mathrm{div}_x\big(A\nabla_x\varphi(f_{sh\tau})\big)\\ &\quad - \partial_\lambda H^{(sh\tau)} - G^{(sh\tau)} - I^{(sh\tau)} = 0,\end{aligned} \tag{4.2}$$

where

$$
\begin{aligned}
&H^{(sh\tau)} = -\varphi'(f_{sh\tau})\big(b'(\lambda)\partial_\lambda\chi_{sh\tau} + M_{sh\tau}\big),\\
&I^{(sh\tau)} = \varphi''(f_{sh\tau})\big\{b'(\lambda)\partial_\lambda f_{sh\tau}\partial_\lambda\chi_{sh\tau} + M_{sh\tau}\partial_\lambda f_{sh\tau} - b'(\lambda)\big|A^{1/2}\nabla_x f_{sh\tau}\big|^2\big\},\\
&G^{(sh\tau)} = \varphi'(f_{sh\tau})\big(R_1^{(sh\tau)} + R_2^{(sh\tau)} + R_3^{(sh\tau)}\big).
\end{aligned}
$$

**Lemma 13.** *Inequality $I^{(sh\tau)} \geqslant 0$ holds in $Q_\tau \times \mathbb{R}_\lambda$.*

**Proof.** We have $b' > 0$, $\varphi''(f_{sh\tau}) \geqslant 0$, as $\varphi$ is convex on $[0,1]$, and $(\partial_\lambda f_{sh\tau})M_{sh\tau} \geqslant 0$, as $f$ is monotone non-decreasing with respect to $\lambda$, and as $M$ is non-negative. Therefore, it suffices to prove that

$$\partial_\lambda f_{sh\tau}\partial_\lambda\chi_{sh\tau} - \big|A^{1/2}\nabla_x f_{sh\tau}\big|^2 \geqslant 0. \tag{4.3}$$

On the strength of items (a)–(c) of the formulation of Problem K, we have

$$\partial_\lambda f_{sh\tau}(\boldsymbol{x},t,\lambda) = \int\limits_{\mathbb{R}^{d+1}_{y,\xi}} \Bigg\{ \int\limits_{\mathbb{R}_\zeta\times\mathcal{L}_q} \omega_{sh\tau}(\boldsymbol{x}-\boldsymbol{y}, t-\xi, \lambda-\zeta)\, d\sigma_{y,\xi}(\zeta,\boldsymbol{q})\Bigg\}\, d\boldsymbol{y}\, d\xi, \tag{4.4}$$

$$\partial_\lambda\chi_{sh\tau} = \int\limits_{\mathbb{R}^{d+1}_{y,\xi}} \Bigg\{ \int\limits_{\mathbb{R}_\zeta\times\mathcal{L}_q} \omega_{sh\tau}(\boldsymbol{x}-\boldsymbol{y}, t-\xi, \lambda-\zeta)|\boldsymbol{q}|^2\, d\sigma_{y,\xi}(\zeta,\boldsymbol{q})\Bigg\}\, d\boldsymbol{y}\, d\xi, \tag{4.5}$$

$$A^{1/2}\nabla_x f_{sh\tau} = -\int\limits_{\mathbb{R}^{d+1}_{y,\xi}} \Bigg\{ \int\limits_{\mathbb{R}_\zeta\times\mathcal{L}_q} \omega_{sh\tau}(\boldsymbol{x}-\boldsymbol{y}, t-\xi, \lambda-\zeta)\boldsymbol{q}\, d\sigma_{y,\xi}(\zeta,\boldsymbol{q})\Bigg\}\, d\boldsymbol{y}\, d\xi. \tag{4.6}$$

Using (4.4)–(4.6), we reduce inequality (4.3) to the equivalent form

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}_{y,\xi}} \Bigg\{ \int\limits_{\mathbb{R}_\zeta\times\mathcal{L}_q} \omega_{sh\tau}|\boldsymbol{q}|^2\, d\sigma_{y,\xi}(\zeta,\boldsymbol{q})\Bigg\}\, d\boldsymbol{y}\, d\xi \int\limits_{\mathbb{R}^{d+1}_{y,\xi}} \Bigg\{ \int\limits_{\mathbb{R}_\zeta\times\mathcal{L}_q} \omega_{sh\tau}\, d\sigma_{y,\xi}(\zeta,\boldsymbol{q})\Bigg\}\, d\boldsymbol{y}\, d\xi\\
&\quad - \Bigg( \int\limits_{\mathbb{R}^{d+1}_{y,\xi}} \Bigg\{ \int\limits_{\mathbb{R}_\zeta\times\mathcal{L}_q} \omega_{sh\tau}\boldsymbol{q}\, d\sigma_{y,\xi}(\zeta,\boldsymbol{q})\Bigg\}\, d\boldsymbol{y}\, d\xi\Bigg)^2 \geqslant 0.
\end{aligned} \tag{4.7}
$$

On the strength of the version of Hölder's inequality (see, for example, [15, Chapter 1, §3, formula (5)]), we conclude that (4.7) holds true. □

On the strength of Lemma 13, we obtain the following inequality from (4.2):

$$
\begin{aligned}
&\partial_t\varphi(f_{sh\tau}) + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\varphi(f_{sh\tau}) - b'(\lambda)\,\mathrm{div}_x\big(A\nabla_x\varphi(f_{sh\tau})\big)\\
&\quad - \partial_\lambda H^{(sh\tau)} - G^{(sh\tau)} \geqslant 0.
\end{aligned} \tag{4.8}
$$

**Step 3.** *Passage to the limit, as $s \searrow 0$.* We have $f_{sh\tau} \to f_{h\tau}$ strongly in $L^r_{\mathrm{loc}}(\mathbb{R}_\lambda \times Q_\tau)$ for any $r \geqslant 1$ and weakly* in $L^\infty(\mathbb{R}_\lambda \times Q_\tau)$, as $s \searrow 0$, due to the well-known properties of mollifying kernels.

**Lemma 14.** *Function $f_{h\tau}(\boldsymbol{x},t,\lambda)$ satisfies in the cylinder $Q_\tau \times \mathbb{R}_\lambda$ the inequality*

$$\partial_t\varphi(f_{h\tau}) + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\varphi(f_{h\tau}) - b'(\lambda)\,\mathrm{div}_x\big(A\nabla_x\varphi(f_{h\tau})\big) \\ - \partial_\lambda H^{(h\tau)} - G^{(h\tau)} \geqslant 0 \tag{4.9}$$

*in the sense of distributions. Here $H^{(h\tau)}$ is a Radon measure in $Q_\tau \times \mathbb{R}_\lambda$, such that*

$$\big\|H^{(h\tau)}\big\|_{C(Q_\tau\times\mathbb{R}_\lambda)^*} \leqslant c, \qquad \mathrm{spt}\,H^{(h\tau)} \subset Q_\tau \times [0,1]_\lambda; \tag{4.10}$$

$$G^{(h\tau)} = \varphi'(f_{h\tau})R_1^{(h\tau)}, \qquad R_1^{(h\tau)} = \mathrm{div}_x\big(a'(\lambda)\boldsymbol{v} f_{h\tau}\big) - \mathrm{div}_x(a'\boldsymbol{v} f)_{h\tau}. \tag{4.11}$$

**Proof.** For any integer non-negative $\alpha$ and $\beta$ the functions $\partial_x^\alpha\partial_t^\beta f_{sh\tau}$ are uniformly bounded with respect to $s$ and converge a.e. in $Q_\tau \times \mathbb{R}_\lambda$ to $\partial_x^\alpha\partial_t^\beta f_{h\tau}$. Hence, we apply the Lebesgue dominated convergence theorem to the sum of the first three terms in (4.8) and conclude that this expression converges in $L^1_{\mathrm{loc}}(Q_\tau \times \mathbb{R}_\lambda)$ to

$$\partial_t\varphi(f_{h\tau}) + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\varphi(f_{h\tau}) - b'(\lambda)\,\mathrm{div}_x\big(A\nabla_x\varphi(f_{h\tau})\big),$$

as $s \searrow 0$. The same arguments give

$$\varphi'(f_{sh\tau})R_1^{(sh\tau)} \to \varphi'(f_{h\tau})R_1^{(h\tau)}$$

in $L^1_{\mathrm{loc}}(Q_\tau \times \mathbb{R}_\lambda)$, as $s \searrow 0$. The passage to the limit in the summands $\partial_\lambda H^{(sh\tau)}$ and $\varphi'(f_{sh\tau})(R_2^{(sh\tau)} + R_3^{(sh\tau)})$ is based on the following lemma.

**Lemma 15.**

(i) *The family of functions $b'(\lambda)\partial_\lambda\chi_{sh\tau} + M_{sh\tau}$ is uniformly bounded in $L^1(Q_\tau \times \mathbb{R}_\lambda)$ with respect to $s$, $h$, and $\tau$.*
(ii) *For any fixed $h,\tau > 0$ function $\chi_{h\tau}(\boldsymbol{x},t,\lambda)$ is Lipschitz continuous on the set $Q_\tau \times \mathbb{R}_\lambda$.*

**Proof.** Let us integrate Eq. (4.1) over the interval $(-\infty,\lambda_0)$ with respect to $\lambda$. Since $\chi$ and $M$ vanish for $\lambda < 0$, we have

$$b'(\lambda_0)\partial_{\lambda_0}\chi_{sh\tau}(\boldsymbol{x},t,\lambda_0) + M_{sh\tau}(\boldsymbol{x},t,\lambda_0) = \Phi^{(sh\tau)}(\boldsymbol{x},t,\lambda_0), \tag{4.12}$$

where

$$\Phi^{(sh\tau)}(\boldsymbol{x},t,\lambda_0) := -\int_{-\infty}^{\lambda_0}\Big(b'(\lambda)\,\mathrm{div}_x(A\nabla_x f_{sh\tau}) - \partial_t f_{sh\tau} \\ - a'(\lambda)\boldsymbol{v}\cdot\nabla_x f_{sh\tau} + R_1^{(sh\tau)} + R_2^{(sh\tau)} + R_3^{(sh\tau)}\Big)\,d\lambda. \tag{4.13}$$

Since $b$ and $\chi_{sh\tau}$ are monotonous non-decreasing with respect to $\lambda$ and since $M_{sh\tau}$ is non-negative, we have that $\Phi^{(sh\tau)} \geqslant 0$. Thus, we get

$$\|b'\partial_{\lambda_0}\chi_{sh\tau} + M_{sh\tau}\|_{L^1(Q_\tau\times\mathbb{R}_\lambda)} = \int_{\mathbb{R}_{\lambda_0}\times Q_\tau} \Phi^{(sh\tau)}(\boldsymbol{x},t,\lambda_0)\,d\boldsymbol{x}\,dt\,d\lambda_0. \tag{4.14}$$

We calculate the right-hand side integral explicitly using integration by parts with respect to $\boldsymbol{x}$ and periodicity property in the terms containing $b'(\lambda)\,\mathrm{div}_x(A\nabla_x f_{sh\tau})$, $a'(\lambda)\boldsymbol{v}\cdot\nabla_x f_{sh\tau}$, $R_1^{(sh\tau)}$, and $R_3^{(sh\tau)}$ (all these integrals are equal to zero), and using integration by parts with respect to $\lambda$ in the term containing $R_2^{(sh\tau)}$. Thus, we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}_{\lambda_0}\times Q_\tau} &\left(\Phi^{(sh\tau)}(\boldsymbol{x},t,\lambda_0)+\left\{b''(\lambda_0)\chi_{sh\tau}-(b''\chi_{h\tau})_s\right\}\right)d\lambda_0\,d\boldsymbol{x}\,dt\\
&=\int\limits_{\mathbb{R}_{\lambda_0}\times\Omega}\int\limits_{-\infty}^{\lambda_0}\left\{f_{sh\tau}(\boldsymbol{x},T-\tau,\lambda)-f_{sh\tau}(\boldsymbol{x},\tau,\lambda)\right\}d\lambda\,d\boldsymbol{x}\,d\lambda_0\\
&\quad+\int\limits_{Q_\tau}\left\{b'(1+s)\chi_{sh\tau}(\boldsymbol{x},t,1+s)-(b'\chi_{h\tau})_s(\boldsymbol{x},t,1+s)\right\}d\boldsymbol{x}\,dt\\
&\quad-\int\limits_{Q_\tau}\left\{b'(-s)\chi_{sh\tau}(\boldsymbol{x},t,-s)-(b'\chi_{h\tau})_s(\boldsymbol{x},t,-s)\right\}d\boldsymbol{x}\,dt.\end{aligned}\tag{4.15}$$

Computing the last two integrals, we take into account that $\lambda\mapsto\chi(\boldsymbol{x},t,\lambda)$, as well as $\lambda\mapsto f(\boldsymbol{x},t,\lambda)$, is a constant function on $(-\infty,0)$ and on $[1,+\infty)$ for fixed $\boldsymbol{x}$ and $t$ and that the support of the regularization kernel $\omega_s(\lambda-\xi)$ lies in the interval $\{\lambda-s\leqslant\xi\leqslant\lambda+s\}$ for any fixed $\lambda$. On the strength of these facts together with the properties $b\in C^2_{\mathrm{loc}}(\mathbb{R})$ and $f\in L^\infty(Q\times\mathbb{R}_\lambda)$, from (4.14) and (4.15) we deduce

$$\|b'\partial_{\lambda_0}\chi_{sh\tau}+M_{sh\tau}\|_{L^1(Q_\tau\times\mathbb{R}_\lambda)}\leqslant c_*,\tag{4.16}$$

where $c_*$ does not depend on $s$, $h$, and $\tau$. Thus, assertion (i) of the lemma is proved.

Now, let us prove that, if $s$ is less than some fixed value $s_*$, then the bound

$$\Phi^{(sh\tau)}(\boldsymbol{x},t,\lambda_0)\leqslant c_{**}(h,\tau)\tag{4.17}$$

holds for any fixed $h,\tau>0$ for all $(\boldsymbol{x},t,\lambda_0)\in\mathbb{R}^d_x\times[0,T]\times\mathbb{R}_\lambda$. Equality (4.13) along with the well-known properties of the mollifying kernels implies the estimate

$$\Phi^{(sh\tau)}(\boldsymbol{x},t,\lambda_0)\leqslant c^{(1)}_{**}(h,\tau)+\left|\int\limits_{-\infty}^{\lambda_0}R_2^{(sh\tau)}\,d\lambda\right|,\tag{4.18}$$

where $c^{(1)}_{**}$ does not depend on $s$, $\boldsymbol{x}$, $t$, and $\lambda_0$. Using Taylor's expansion

$$b'(\lambda_0)-b'(\xi)=b''(\xi)(\lambda_0-\xi)+\rho(\lambda_0,\xi),\qquad\left|\rho(\lambda_0,\xi)\right|\leqslant c_\rho|\lambda_0-\xi|^2,\tag{4.19}$$

we represent

$$\begin{aligned}\int\limits_{-\infty}^{\lambda_0}R_2^{(sh\tau)}\,d\lambda&=b'(\lambda_0)\partial_{\lambda_0}\chi_{sh\tau}-(b'\partial_{\lambda_0}\chi_{h\tau})_s\\
&=\int\limits_0^1\bar\rho(\xi)\left(\chi_{h\tau}(\boldsymbol{x},t,\lambda_0)-\chi_{h\tau}(\boldsymbol{x},t,\lambda_0-s\xi)\right)d\xi\end{aligned}$$

$$+\int\limits_{\mathbb{R}_\xi} \omega_s(\lambda_0-\xi)\rho(\lambda_0,\xi)\chi_{h\tau}(\boldsymbol{x},t,\xi)\,d\xi, \tag{4.20}$$

where $\bar\rho(\lambda)=\omega'(\lambda)\lambda+\omega(\lambda)$, $\int_{\mathbb{R}_\lambda}\bar\rho(\lambda)\,d\lambda=0$. As $\operatorname{spt}\omega\subset[0,1]$, $\max\omega=1$, and formulas (1.11c) and (1.11d) take place, the bound

$$\chi_{h\tau}(\boldsymbol{x},t,\lambda)\leqslant\frac{1}{h^d}\frac{1}{\tau}\int\limits_{|y-x|\leqslant h}\int\limits_{|\zeta-t|\leqslant\tau}\chi(\boldsymbol{y},\zeta,\lambda)\,d\boldsymbol{y}\,d\zeta\leqslant c_{**}^{(2)}(h,\tau) \tag{4.21}$$

is valid for all $(\boldsymbol{x},t,\lambda)\in\mathbb{R}_x^d\times[0,T]\times\mathbb{R}_\lambda$ with a constant $c_{**}^{(2)}$ that does not depend on $\boldsymbol{x}$, $t$, and $\lambda$. Also, in view of (4.19) we observe that

$$\left|\int\limits_{\mathbb{R}_\xi}\omega_s(\lambda_0-\xi)\rho(\lambda_0,\xi)\chi_{h\tau}(\boldsymbol{x},t,\xi)\,d\xi\right|\leqslant c_\rho\,s\int\limits_{-s}^{s}\chi_{h\tau}(\boldsymbol{x},t,\xi)\,d\xi. \tag{4.22}$$

Aggregating (4.18), (4.20)–(4.22), we conclude that (4.17) holds true.

Next, integrating both sides of (4.12) with respect to $\lambda_0$ over the interval $[\lambda',\lambda'']$ ($\lambda'<\lambda''$), we derive

$$\chi_{sh\tau}(\boldsymbol{x},t,\lambda'')-\chi_{sh\tau}(\boldsymbol{x},t,\lambda')=\int\limits_{\lambda'}^{\lambda''}\frac{\Phi^{(sh\tau)}(\boldsymbol{x},t,\lambda_0)\,d\lambda_0}{b'(\lambda_0)}-\int\limits_{\lambda'}^{\lambda''}\frac{M_{sh\tau}(\boldsymbol{x},t,\lambda_0)\,d\lambda_0}{b'(\lambda_0)}.$$

As $\chi_{sh\tau}$ is monotone with respect to $\lambda$, $M_{sh\tau}$ is non-negative, $b'>0$, and (4.17) holds, we conclude that

$$0\leqslant\chi_{sh\tau}(\boldsymbol{x},t,\lambda'')-\chi_{sh\tau}(\boldsymbol{x},t,\lambda')\leqslant\left(\frac{c_{**}(h,\tau)}{\min_{\lambda_0\in[0,1]}b'(\lambda_0)}\right)|\lambda''-\lambda'|. \tag{4.23}$$

Passing to the limit, as $s\searrow0$, from (4.23) we obtain

$$0\leqslant\chi_{h\tau}(\boldsymbol{x},t,\lambda'')-\chi_{h\tau}(\boldsymbol{x},t,\lambda')\leqslant\left(\frac{c_{**}(h,\tau)}{\min_{\lambda_0\in[0,1]}b'(\lambda_0)}\right)|\lambda''-\lambda'|,$$

which completes the proof of assertion (ii). □

Assertion (i) of Lemma 15 immediately implies the bound

$$\left\|H^{(sh\tau)}\right\|_{L^1(\mathbb{R}_\lambda\times Q_\tau)}\leqslant\max_{\kappa\in[0,1]}\left|\varphi'(\kappa)\right|c_*=c, \tag{4.24}$$

where $c$ does not depend on $s$, $h$, and $\tau$. Thus, $H_{sh\tau}\to H_{h\tau}$ weakly* in $C(\mathbb{R}_\lambda\times Q_\tau)^*$, as $s\searrow0$, the bound (4.10) holds true, and the support of $H_{h\tau}$ lies in the strip $\{0\leqslant\lambda\leqslant1\}$, as the supports of both $\partial_\lambda\chi$ and $M$ lie there.

Now, in order to complete the verification of Lemma 14, it suffices to prove the following lemma.

**Lemma 16.** *For any $\zeta\in C_0^\infty(R_\lambda\times Q_\tau)$,*

$$\int\limits_{\mathbb{R}_\lambda\times Q_\tau}\zeta\varphi'(f_{sh\tau})\left(R_2^{(sh\tau)}+R_3^{(sh\tau)}\right)d\boldsymbol{x}\,dt\,d\lambda\to0,\quad \textit{as } s\searrow0.$$

**Proof.** We have

$$\begin{aligned}
&\int\limits_{Q_\tau\times\mathbb{R}_\lambda} \zeta\varphi'(f_{sh\tau})\big(R_2^{(sh\tau)}+R_3^{(sh\tau)}\big)\,d\boldsymbol{x}\,dt\,d\lambda\\
&\quad=\int\limits_{Q_\tau\times\mathbb{R}_\lambda} \big\{\mathrm{div}_x\big(A\nabla_x\big(\zeta\varphi'(f_{sh\tau})\big)\big)F^s(\boldsymbol{x},t,\lambda)\\
&\qquad+\big[\varphi'(f_{sh\tau})\partial_\lambda\zeta+\zeta\varphi''(f_{sh\tau})\partial_\lambda f_{sh\tau}\big]\Psi^s(\boldsymbol{x},t,\lambda)\big\}\,d\boldsymbol{x}\,dt\,d\lambda,
\end{aligned}$$

where

$$F^s(\boldsymbol{x},t,\lambda)=b'f_{sh\tau}-(b'f_{h\tau})_s,\qquad \Psi^s(\boldsymbol{x},t,\lambda)=\int\limits_{-\infty}^{\lambda} R_2^{(sh\tau)}(\boldsymbol{x},t,\tilde\lambda)\,d\tilde\lambda.$$

Recall that

$$0\leqslant f_{sh\tau}\leqslant 1\quad\text{and}\quad \int\limits_{-\infty}^{\infty}\big|\partial_\lambda f_{sh\tau}(\boldsymbol{x},t,\lambda)\big|\,d\lambda=1.$$

Hence,

$$\begin{aligned}
&\Bigg|\int\limits_{Q_\tau\times\mathbb{R}_\lambda} \zeta\varphi'(f_{sh\tau})\big(R_2^{(sh\tau)}+R_3^{(sh\tau)}\big)\,d\boldsymbol{x}\,dt\,d\lambda\Bigg|\\
&\quad\leqslant c(\zeta,\varphi,\tau,h)\sup_{(x,t,\lambda)\in\mathrm{spt}\,\zeta}\big(\big|\Psi^s(\boldsymbol{x},t,\lambda)\big|+\big|F^s(\boldsymbol{x},t,\lambda)\big|\big).
\end{aligned}$$

This estimate implies that it is sufficient to prove that $F^s\to 0$ and $\Psi^s\to 0$ uniformly on any compact subset of $Q_\tau\times\mathbb{R}_\lambda$, as $s\searrow 0$. The first of these limiting relations follows from the representation

$$F^s(\boldsymbol{x},t,\lambda)=\int\limits_{\mathbb{R}_\xi}\omega_s(\lambda-\xi)\big(b'(\lambda)-b'(\xi)\big)f_{h\tau}(\boldsymbol{x},t,\xi)\,d\xi,$$

the bound $|f_{h\tau}|\leqslant 1$, and Taylor's expansion (4.19). The second limiting relation follows from (4.20), (4.22), and the fact that, on the strength of assertion (ii) of Lemma 15, the limiting relations

$$\int\limits_0^1\bar\rho(\xi)\big(\chi_{h\tau}(\boldsymbol{x},t,\lambda)-\chi_{h\tau}(\boldsymbol{x},t,\lambda-s\xi)\big)\,d\xi\to 0\quad\text{and}$$

$$c_\rho s\int\limits_{-s}^{s}\chi_{h\tau}(\boldsymbol{x},t,\xi)\,d\xi\to 0$$

hold true, as $s\searrow 0$. □

□

**Step 4.** *Passage to the limit, as $\tau \searrow 0$.*

**Lemma 17.** *Function $f_h(\boldsymbol{x}, t, \lambda)$ satisfies the integral inequality*

$$\int\limits_{Q\times\mathbb{R}_\lambda} \varphi(f_h)\bigl\{\partial_t\zeta + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\zeta + b'(\lambda)\operatorname{div}_x(A\nabla_x\zeta)\bigr\}\,d\boldsymbol{x}\,dt\,d\lambda + \int\limits_{\Omega\times\mathbb{R}_\lambda} \varphi(f_{0h})\zeta(\boldsymbol{x},0,\lambda)\,d\boldsymbol{x}\,d\lambda - \int\limits_{Q\times\mathbb{R}_\lambda} \partial_\lambda\zeta\,dH^{(h)}(\boldsymbol{x},t,\lambda) + \int\limits_{Q\times\mathbb{R}_\lambda} \zeta G^{(h)}\,d\boldsymbol{x}\,dt\,d\lambda \leqslant 0, \tag{4.25}$$

*where $\zeta(\boldsymbol{x}, t, \lambda)$ is a non-negative 1-periodic in $\boldsymbol{x}$ smooth function vanishing for $t = T$ and sufficiently large $\lambda$; $H^{(h)}$ is a Radon measure in $Q \times \mathbb{R}_\lambda$ such that*

$$\bigl\|H^{(h)}\bigr\|_{C(Q\times\mathbb{R}_\lambda)^*} \leqslant c_2, \qquad \operatorname{spt} H^{(h)} \subset Q \times [0,1]_\lambda, \tag{4.26}$$

$$G^{(h)} = \varphi'(f_h)\bigl\{\operatorname{div}_x\bigl(a'(\lambda)\boldsymbol{v} f_h\bigr) - \operatorname{div}_x\bigl(a'(\lambda)\boldsymbol{v} f\bigr)_h\bigr\}. \tag{4.27}$$

**Proof.** Fix an arbitrary small $t_0 \in (0, T)$ and assume that $\tau < t_0$. Since $\partial_x^\alpha f_{h\tau}$ converges to $\partial_x^\alpha f_h$ in $L^r_{\mathrm{loc}}(Q_{t_0} \times \mathbb{R}_\lambda)$ for any integer non-negative $\alpha$ and for any $r \geqslant 1$, and since $f_{h\tau}(t_0)$ converges to $f_h(t_0)$ in $L^r_{\mathrm{loc}}(\Omega \times \mathbb{R}_\lambda)$ for almost every $t_0$, we conclude that $G^{(h\tau)} \to G^{(h)}$ in $L^1_{\mathrm{loc}}(Q_{t_0} \times \mathbb{R}_\lambda)$ and $\varphi(f_{h\tau}(t_0)) \to \varphi(f_h(t_0))$ in $L^1_{\mathrm{loc}}(\Omega \times \mathbb{R}_\lambda)$, as $\tau \searrow 0$.

Passing to the limit in (4.9), as $\tau \searrow 0$, we derive the inequality

$$\int\limits_{Q_{t_0}\times\mathbb{R}_\lambda} \varphi(f_h)\bigl\{\partial_t\zeta + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\zeta + b'(\lambda)\operatorname{div}_x(A\nabla_x\zeta)\bigr\}\,d\boldsymbol{x}\,dt\,d\lambda + \int\limits_{\Omega\times\mathbb{R}_\lambda} \varphi\bigl(f_h(\boldsymbol{x},t_0,\lambda)\bigr)\zeta(\boldsymbol{x},t_0,\lambda)\,d\boldsymbol{x}\,d\lambda - \int\limits_{Q_{t_0}\times\mathbb{R}_\lambda} \partial_\lambda\zeta\,dH^{(h)} + \int\limits_{Q_{t_0}\times\mathbb{R}_\lambda} \zeta G^{(h)}\,d\boldsymbol{x}\,dt\,d\lambda \leqslant 0, \tag{4.28}$$

where measure $H^{(h)}$ is the weak* limit in $C(Q \times \mathbb{R}_\lambda)^*$ of $H^{(h\tau)}$ and $\zeta(\boldsymbol{x}, t, \lambda)$ is an arbitrary 1-periodic in $\boldsymbol{x}$ smooth test function which vanishes for $t \in [T - t_0, T]$.

Recall that $\varphi$ is a continuous convex on $[0, 1]$ function and that $f_h(t_0)$ converges weakly* in $L^\infty(\Omega \times \mathbb{R}_\lambda)$ to $f_{0h}$, due to Lemma 11. On the strength of the lower semicontinuity property, this yields

$$\liminf_{t_0\searrow 0} \int\limits_{\Omega\times\mathbb{R}_\lambda} \zeta(t_0)\varphi\bigl(f_h(t_0)\bigr)\,d\boldsymbol{x}\,d\lambda \geqslant \int\limits_{\Omega\times\mathbb{R}_\lambda} \zeta(0)\varphi(f_{0h})\,d\boldsymbol{x}\,d\lambda.$$

Using this inequality and passing to the limit in (4.28), as $t_0 \searrow 0$, we obtain (4.25). □

**Step 5.** *Passage to the limit, as $h \searrow 0$.* In view of the properties of the mollifying kernels, we have

$$\varphi(f_h) \to \varphi(f) \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}_\lambda \times Q), \qquad \varphi(f_{0h}) \to \varphi(f_0) \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}_\lambda \times \Omega) \tag{4.29}$$

$\forall p \geqslant 1$, as $h \searrow 0$. On the strength of the bound (4.26), the limiting relation

$$H^{(h)} \to H_\varphi \quad \text{weakly* in } C(R_\lambda \times Q)^* \tag{4.30}$$

takes place. On the strength of the properties of $H^{(h)}$ stated in Lemma 17, measure $H_\varphi$ has the properties from the formulation of Proposition 6. Finally,

$$G^{(h)} \to 0 \quad \text{in } L^1_{\mathrm{loc}}(Q \times \mathbb{R}_\lambda), \text{ as } h \searrow 0, \tag{4.31}$$

on the strength of [4, Lemma II.1] and (4.29). Using (4.29)–(4.31), we fulfill the limiting transition, as $h \searrow 0$, in (4.25) and obtain (1.14). Theorem 6 is proved.

## 5. Proof of Theorem 7

Proof of the first part of the theorem, i.e., of the assertion on the structure of solutions of Problem K, is based on the special choice of test functions $\zeta$ and $\varphi$ in the renormalized inequality (1.14): we take

$$\varphi(f) = f(f-1), \tag{5.1}$$

and $\zeta(\boldsymbol{x}, t, \lambda) = \zeta_1(\lambda)\zeta_2(t)$ such that

$$\zeta_1 \text{ is non-negative} \quad \text{and} \quad \zeta_1 = 1 \quad \text{on } [0,1], \tag{5.2}$$

$$\zeta_2 \text{ is non-negative}, \quad \zeta_2(T) = 0 \quad \text{and} \quad \zeta_2' < 0 \quad \text{for } t < T. \tag{5.3}$$

It is easy to see that such choice of $\varphi$ and $\zeta$ makes sense. Substituting these functions into (1.14) and observing that $\nabla_x \zeta = 0$, $\varphi(f_0) = 0$, and that

$$\int\limits_{Q \times \mathbb{R}_\lambda} \partial_\lambda \zeta \, dH_\varphi(\boldsymbol{x}, t, \lambda) = 0$$

(due to (5.2) and spt $H \subset (Q \times [0,1]_\lambda)$), we see that (1.14) takes the form

$$\int\limits_{Q \times [0,1]_\lambda} \zeta_1 \varphi(f) \partial_t \zeta_2 \, d\boldsymbol{x}\, dt\, d\lambda \leqslant 0.$$

In view of (5.1), (5.3), and point (a) of the formulation of Problem K, this inequality implies $\varphi(f) \equiv 0$, which yields that $f(\boldsymbol{x}, t, \lambda)$ attains one of only two values, either zero or one, at almost every point $(\boldsymbol{x}, t, \lambda) \in [0,1] \times Q$.

The second assertion of the theorem is true due to Remark 2 and to the first assertion of the theorem.

We end our paper by the remark that, since $f$ is the distribution function of the Young measure $\mu_{x,t}$ and since $f(\boldsymbol{x}, t, \lambda) = 0$ for $\lambda < u(\boldsymbol{x}, t)$ and $f(\boldsymbol{x}, t, \lambda) = 1$ for $\lambda \geqslant u(\boldsymbol{x}, t)$ due to Theorem 7 and Remark 3, we have that $\mu_{x,t}$ is the Dirac measure on $\mathbb{R}_\lambda$ centered at the point $\lambda = u(\boldsymbol{x}, t)$ for a.e. $(\boldsymbol{x}, t) \in Q$. On the strength of [11, Chapter 3, Theorem 2.31], this yields that the sequence of solutions $u_\varepsilon$ of problem (1.8), (1.1b), (1.1c) converges to the entropy solution $u$ strongly in $L^1$, as $\varepsilon \searrow 0$.

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