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A Cauchy problem for the Tartar equation

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In order to study weak limits of quadratic expressions of oscillatory solutions of partial differential equations, there was proposed a construction of H-measures defined on the space of positions and frequencies. The present paper is devoted to the investigation of the *Tartar equation*

$$\partial_t \mu_t + \sum_{i=1}^{2} v_i \partial_{x_i} \mu_t + \sum_{i,j=1}^{2} \partial_y (\mu_t Y_{ij} \partial_{x_i} v_j) = 0,$$

which describes the evolution of H-measure μ_t associated with a sequence of oscillatory solutions of the linear transport equation

$$\partial_t \rho + \sum_{i=1}^{2} v_i \partial_{x_i} \rho = 0$$

in cases when a given solenoidal velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ is sufficiently smooth. Here, $(t, \boldsymbol{x}, y) \in (0, T) \times \Omega \times S^1$, $0 < T < +\infty$, Ω is a bounded open subset of $\mathbb{R}^2$ and S^1 is the unit circle in $\mathbb{R}^2$, given coefficients $Y_{ij} = Y_{ij}(y)$ are infinitely smooth.

Assuming that $\boldsymbol{v}$ belongs to $L_2(0, T; H_0^1(\Omega))$ we establish the well posedness of Cauchy problem for the Tartar equation in the same measure class as the H-measures are in. For this purpose, we develop and use an extension of the theory of Lagrange coordinates for a case of non-smooth solenoidal velocity fields.

1 Introduction

1.1 Foreword

The linear transport equation of the form

$$\partial_t \rho(\boldsymbol{x}, t) + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \nabla_x \rho(\boldsymbol{x}, t) = 0 \tag{1.1}$$

is a part of a large variety of mathematical models of continuum mechanics. Once (1.1) allows for a sequence of highly oscillatory solutions, there arises a question of studying an evolution of such oscillations. Assuming, in line with later considerations in the present paper, that a sequence of solutions $\{\rho_\varepsilon(\boldsymbol{x},t)\}_{\varepsilon>0}$ is defined in the cylinder $\Omega\times[0,T]$ ($\Omega\subset\mathbb{R}^2$, $T<+\infty$) and tends to some limit ρ^* weak* in $L_\infty(\Omega\times[0,T])$ as $\varepsilon\to 0$, this question amounts to asking *to describe the set of weak limits* $f^*(\boldsymbol{x},t) = w\text{-}\lim\limits_{\varepsilon\to 0} f(\rho_\varepsilon(\boldsymbol{x},t))$ *for all continuous* f.

There does not exist a universal method for the investigation of all problems like this. Each of the existing approaches covers a certain class of situations. The focus of the present work is on *H-measures*, proposed by Tartar in [26] and by Gerárd in [10] (Gerárd called these objects *microlocal defect measures*). H-measures contain information about weak limits of sequences $\{\rho_\varepsilon\varphi_1\mathcal{A}[\varphi_2\rho_\varepsilon]\}$, where $\mathcal{A}$ is an arbitrary pseudodifferential operator of zero order and φ_1, φ_2 are arbitrary compactly supported continuous in Ω functions. It is shown in [10], [26, §4.2] and [27] that this tool can be successfully applied for studying the limit regimes appearing as $\varepsilon\to 0$ within the frameworks of models describing a motion of continuous media having either small asymptotic or shear structures. Also, a set of compactness results can be obtained by the systematic use of H-measures (as in [17]–[19]) for sequences of oscillatory solutions of quasilinear hyperbolic equations of the first order.

Whenever an object like H-measures is used in problems concerning oscillations, it is strongly desirable to describe its evolution on a macroscopic level. Usually this means *to obtain an evolutionary equation independent of* ε*, such that the considered object solves it, and thus can be determined directly from the data given at the time* $t=0$ *without shifting to the microscopic level and dealing with the sequence* $\{\rho_\varepsilon(\boldsymbol{x},t)\}$.

The present paper is devoted to the analysis of the equation that describes the evolution of H-measures associated with a sequence of solutions of (1.1). Here, it is called the Tartar equation, in line with the original idea on transport properties of H-measures that was proposed and developed by Tartar in [26, §3]. The precise notions of H-measures and the Tartar equation will be given in §1.2. Our research is somewhat similar to that involving the use of the concepts of Young measures and Wigner distributions. These tools are very similar to H-measures and are used to answer questions concerning the appearance of oscillatory solutions of partial differential equations (PDEs). In many cases, Young measures [32] provide the best means of investigating the behaviour of weakly convergent sequences under superpositions of nonlinearities, as they may have a good structure and even lead to the identity $f^*(\boldsymbol{x},t) = f(\rho^*(\boldsymbol{x},t))$ a.e. in $\Omega\times[0,T]$. Examples of such situations and the corresponding transport properties of Young measures can be found in [5], [7], [14]–[16], [24]–[27]. Wigner distributions [31] are intended, in particular, for holding information about weak continuity properties of quadratic forms on spaces of solutions of linear hyperbolic systems [3]. Therefore, Wigner distributions are a good tool for analyzing the transport of wave energy density in the cases when such systems describe wave motions possessing high-frequency asymptotics [20].

We end our foreword by outlining the notations of the paper. From now on, we suppose that Ω is a bounded open subset of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, $Q_T=\Omega\times[0,T]$, $0<T<+\infty$; S^1 is the unit circle in $\mathbb{R}^2$. We denote $\partial_i=\partial/\partial x_i$ ($i=1,2$), $\partial_t=\partial/\partial t$,

$\partial y = \partial/\partial y$, where $\boldsymbol{x} = (x_1, x_2)$, t, and y are elements of the sets Ω, $[0,T]$, and S^1, respectively, $\nabla_x a = (\partial_1 a, \partial_2 a)$; $\nabla_x \boldsymbol{a} = \|\partial_i a_j\|_{i,j=1,2}$ and $\mathrm{div}_x \boldsymbol{a} = \partial_1 a_1 + \partial_2 a_2$. If $\boldsymbol{a}$ satisfies $\mathrm{div}_x \boldsymbol{a} = 0$, then $\boldsymbol{a}$ is called a *solenoidal* vector. $a \circ b$ denotes that function b is under superposition of functions a; the same notation is used for operators. $A : B$ represents the sum

$$\sum_{i,j=1}^{2} A_{ij} B_{ij}$$

and $(\varphi_1 * \varphi_2)(\boldsymbol{x}) = \int_{\mathbb{R}^2} \varphi_1(\boldsymbol{x} - \boldsymbol{y})\varphi_2(\boldsymbol{y})d\boldsymbol{y}$ is a convolution of the functions φ_1 and φ_2. Functions defined merely on Ω and undergoing integration with respect to $\boldsymbol{x}$ over $\mathbb{R}^2$ are supposed to be extended outside Ω by zero.

The other notations in the paper either do not differ from well-known and commonly accepted ones or are to be introduced as soon as it is necessary.

1.2 Notions of H-measures and the Tartar equation

The definition of H-measures is based on the following fundamental theorem [26, theorem 1.1 and corollary 1.2].

Theorem H (Existence of H-measures). Let $U_\varepsilon \to 0$ in $L_2(\Omega)$ weakly as $\varepsilon \to 0$. Then, after extracting a subsequence (for which the index ε is preserved), there exists a non-negative Borel measure μ on $\Omega \times S^1$, such that, for all compactly supported (in Ω) continuous functions φ_1, φ_2, and every pseudodifferential operator $\mathcal{A}:\ L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ of zero order, one has

$$\langle \mu,\ a\varphi_1\varphi_2 \rangle = \lim_{\varepsilon\to 0} \int_\Omega \varphi_1 U_\varepsilon \mathcal{A}[\varphi_2 U_\varepsilon] d\boldsymbol{x}, \tag{1.2}$$

where $a \in C(S^1)$ is the principal symbol of $\mathcal{A}$.

We remark that the linear span of the set $\{a\varphi_1\varphi_2\}$ is dense in $C_0(\Omega) \times C(S^1)$. Recall that in terms of the Fourier transform F,

$$F[u](\boldsymbol{\xi}) = \int_{\mathbb{R}^2} \exp(2\pi i \boldsymbol{x} \cdot \boldsymbol{\xi}) u(\boldsymbol{x}) d\boldsymbol{x},$$

the operator $\mathcal{A}$ is defined by the formula $F[\mathcal{A}[u]](\boldsymbol{\xi}) = a(\boldsymbol{\xi}/|\boldsymbol{\xi}|)F[u](\boldsymbol{\xi})$. Thus, due to Parseval's theorem, the identity (1.2) has the form

$$\langle \mu,\ a\varphi_1\varphi_2 \rangle = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} F[\varphi_1 U_\varepsilon](\boldsymbol{\xi})\, a(\boldsymbol{\xi}/|\boldsymbol{\xi}|) \overline{F[\varphi_2 U_\varepsilon](\boldsymbol{\xi})} d\boldsymbol{\xi}$$

Here, the bar denotes complex conjugation.

Definition 1.1. The measure μ is called the *H-measure associated with the extracted subsequence of* $\{U_\varepsilon\}$.

Now, suppose that coefficients v_1, v_2 in (1.1) are in $C^1(Q_T)$ and a sequence $\rho_\varepsilon(\boldsymbol{x},t)$ of solutions of (1.1) converges in $L_\infty(Q_T)$ weak* to a limit $\rho^*(\boldsymbol{x},t)$ as $\varepsilon \to 0$. The supposition of existence of such a sequence makes sence because of the theory of linear transport equations (for details see lemma 2.1). Consider a family of H-measures $\{\mu_t\}$ associated with the extracted subsequence of $\{\rho_\varepsilon - \rho^*\}$, depending on t. Evidently, μ_t is defined for almost every $t \in [0,T]$. Similarly to [26, Theorem 3.4], one establishes that μ_t solves the equation

$$\int_0^T \langle \mu_t,\, \partial_t\Phi + \{v_1\xi_1 + v_2\xi_2\,,\, \Phi\} + \Phi \mathrm{div}_x \boldsymbol{v}\rangle dt + \langle \mu_t|_{t=0}, \Phi|_{t=0}\rangle = 0, \tag{1.3}$$

where $(t,\boldsymbol{x},\boldsymbol{\xi}) \in [0,T]\times\Omega\times\mathbb{R}^2$, $\Phi = \Phi(t,\boldsymbol{x},\boldsymbol{\xi}/|\boldsymbol{\xi}|)$ is an arbitrary test function of a class $C^1([0,T]\times\Omega\times S^1)$, satisfying $\Phi|_{t=T} = 0$ and the Poisson bracket has the form $\{\alpha,\beta\} = \nabla_\xi\alpha\cdot\nabla_x\beta - \nabla_x\alpha\cdot\nabla_\xi\beta$. We remark that differentiating a test function Φ with respect to ξ_i $(i = 1,2)$ does not output integrands off the domain of definition of measure μ_t because the Poisson bracket in (1.3) is continuous in $[0,T]\times\Omega\times S^1$ and homogeneous of zero order with respect to the variable $\boldsymbol{\xi}$.

If we parametrize the unit circle S^1 by means of the angular coordinate y, so that $S^1 = \{y (\mathrm{mod}\, 2\pi)\}$, and change variables ξ_1, ξ_2 to y, r by the formulae $\xi_1 = r\cos y$, $\xi_2 = r\sin y$, where r is the radial coordinate on the plain, then (1.3) takes the form

$$\int_0^T \langle \mu_t,\, \partial_t\Phi + \mathrm{div}_x(\Phi\boldsymbol{v}) + (Y : \nabla_x\boldsymbol{v})\partial_y\Phi\rangle dt + \langle \mu_t|_{t=0}, \Phi|_{t=0}\rangle = 0, \tag{1.4}$$

where

$$Y = \begin{pmatrix} -\frac{1}{2}\sin 2y & \cos^2 y \\ -\sin^2 y & \frac{1}{2}\sin 2y \end{pmatrix}$$

and $\Phi = \Phi(t,\boldsymbol{x},y)$ is a test function satisfying $\Phi \in C^1([0,T]\times\Omega\times S^1)$, $\Phi|_{t=T} = 0$.

In the sense of the theory of distributions, equation (1.4) is equivalent to the linear partial differential equation

$$\partial_t\mu_t + \boldsymbol{v}\cdot\nabla_x\mu_t + \partial_y(\mu_t Y : \nabla_x\boldsymbol{v}) = 0. \tag{1.5}$$

Definition 1.2. Equation (1.5) is called the *Tartar equation.*

If a velocity field $\boldsymbol{v} = (v_1,v_2)$ in (1.1) is nonsmooth, i.e. in $L_2(0,T;W_2^2(\Omega))$, then the relevant equation (1.5) takes place for H-measures as well. This fact was rigorously obtained in [22] in view of a problem of the motion of a nonhomogeneous viscous incompressible fluid [1] in a case of absolute values of highly oscillatory distributions of density being uniformly bounded in Ω.

Besides [22, §2], the H-measure μ_t is absolutely continuous with respect to the Lebesgue measure on Ω, and, as a functional on $C(\Omega\times S^1)$, admits natural expansion onto $L_2(\Omega, C(S^1))$. Due to the Lebesgue — Nikodym theorem [2, ch. V, §5.5], these properties imply that the H-measure μ_t is a natural continuation of some mapping $\nu_t \in L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1))$ into the space of Borel measures on $\Omega\times S^1$. That is, for any function $\varphi \in L_2(\Omega, C(S^1))$ for a.e. $t \in [0,T]$ one has

$$\langle \mu_t, \varphi\rangle = \int_\Omega \langle \nu_{t,x}, \varphi(\boldsymbol{x},\cdot)\rangle d\boldsymbol{x}. \tag{1.6}$$

Thus, we denote

$$d\mu_t(\boldsymbol{x}, y) = d\boldsymbol{x}\, d\nu_{t,x}(y).$$

NOTATION 1.3. In the above formulations we have introduced $L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1))$, which is the space of weakly measurable (with respect to the Lebesgue measure on Ω) mappings $\boldsymbol{x} \to \lambda_x$ of Ω into $\mathcal{M}_+(S^1)$ equipped with the norm

$$\|\lambda\|_{L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1))} = \left(\int_\Omega \|\lambda_x\|^2 d\boldsymbol{x}\right)^{1/2} \quad \forall \lambda \in L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1)).$$

$\mathcal{M}_+(S^1)$ is the set consisting of non-negative measures from the dual space of $C(S^1)$ and $\|\cdot\|$ is the norm in $\mathcal{M}_+(S^1)$ defined by the formula $\|\lambda\| = \langle \lambda,\, 1\rangle\ \forall \lambda \in \mathcal{M}_+(S^1)$ (for details, see [2, ch.III, §1.6]).

1.3 Main results

In line with the topics of the previous paragraph, there arises the question of finding the minimum regularity conditions on $(v_1,\, v_2)$ providing the well posedness in the class of non-negative Borel measures on $\Omega \times S^1$ of Cauchy problems for (1.5), with initial data $\mu_t|_{t=0} = \mu_0$, such that $d\mu_0(\boldsymbol{x}, y) = d\boldsymbol{x} d\nu_{0,x}(y)$, $\nu_0 \in L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1))$. In the present paper, we give an answer to this question in the case of solenoidal velocity fields $\boldsymbol{v}(\boldsymbol{x}, t)$.

THEOREM 1.4 (On the well-posedness of Cauchy problem for the Tartar equation.) *If $\boldsymbol{v} \in L_2(0, T; \dot{J}^1(\Omega))$ and the non-negative measure μ_0 defined on $\Omega \times S^1$ is such that $d\mu_0(\boldsymbol{x}, y) = d\boldsymbol{x} d\nu_{0,x}(y)$, $\nu_0 \in L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1))$, then the Cauchy problem for* (1.5)*, with Cauchy data $\mu_t|_{t=0} = \mu_0$, has a unique non-negative solution μ_t such that $d\mu_t(\boldsymbol{x}, y) = d\boldsymbol{x} d\nu_{t,x}(y)$, $\nu \in L_2(0, T; L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1)))$.*

NOTATION 1.5. $\dot{J}^1(\Omega)$, $J(\Omega)$ are the closures in $W_2^1(\Omega)$ and $L_2(\Omega)$, respectively, of the set of infinitely smooth solenoidal vector functions, compactly supported in Ω.

The main obstacle in the justification of this theorem is as follows. Rewriting (1.5) in the equivalent form $\partial_t \mu_t + \mathrm{div}_{x,y}(\mu_t V) = 0$, where $V = \{v_1,\, v_2,\, Y : \nabla_x \boldsymbol{v}\}$, it is easy to observe that V is not essentially bounded in $[0, T] \times \Omega \times S^1$. Hence it is impossible to use the well-known technical method [6] based on use of Grownwall's lemma for establishing *a priori* estimates for solutions of (1.5). Therefore, the original idea on verification of theorem 1.4 consists of the use of the Lagrangian representation of (1.5) and is based on an assumption that more simple forms of the equation would provide a way to overcome the aforementioned obstacle. However, the legitimacy of changing Eulerian variables to Lagrangian ones was justified for velocity fields $\boldsymbol{v}(\boldsymbol{x}, t)$ that were at least in $L_1(0, T; W_2^2(\Omega))$ ($\Omega \subset \mathbb{R}^3$) [9], and the question of less smooth $\boldsymbol{v}(\boldsymbol{x}, t)$ was still open. In order to complete this case we propose the concept of *Lagrange transforms*, which extends the theory of Lagrange coordinates into the case of solenoidal velocity fields being in $L_\gamma(0, T; W_\alpha^1(\Omega))$.

Lagrange transforms appear by virtue of the *Lagrange operator*, which we define as follows. For any fixed $t \in [0,T]$ and for a function $f(\cdot,t) \in L_p(\Omega)$, denote a solution of the Cauchy problem

$$Q_T: \ \partial_s F^{(t)} + \operatorname{div}_x(\boldsymbol{v}(\boldsymbol{x},s)F^{(t)}) = 0, \quad \Omega: \ F^{(t)}(\boldsymbol{x},s)|_{s=t} = f(\boldsymbol{x},t)$$

by $F^{(t)}(\boldsymbol{x},t)$. Recall that if $f \in L_\vartheta(0,T;L_p(\Omega))$ then, for a.e. $t \in [0,T]$, there exists a unique solution $F^{(t)}(\boldsymbol{x},s)$, belonging, as a function of $\boldsymbol{x}$, s, to either $C([0,T];L_p(\Omega))$ (in the case $p<\infty$) or $L_\infty(Q_T) \cap C([0,T];L_{p'}(\Omega))$, with $p'<\infty$ arbitrary (in the case $p=\infty$) [6, corollaries II.1, II.2].

Definition 1.6. The *Lagrange operator* $\mathcal{L}: L_p(\Omega) \to L_p(\Omega)$ associated to a velocity field $\boldsymbol{v}$, is defined by $\mathcal{L}[f](\boldsymbol{x},t) = F^{(t)}(\boldsymbol{x},0), \quad t \in [0,T]$.

Definition 1.7. $\mathcal{L}[f]$ is called the *Lagrange transform* of f.

We will prove in §2 that the operator $\mathcal{L}$ has the following properties.

Proposition 1.8. *If $f \in L_\vartheta(0,T;L_p(\Omega))$, $\boldsymbol{v} \in L_\gamma(0,T;\overset{\circ}{W}{}^1_\alpha(\Omega) \cap J(\Omega))$, then we have the following.*

(1) *$\mathcal{L}[f](\boldsymbol{x},t)$ is measurable in Q_T.*

(2) *$\mathcal{L}[f] \in L_\vartheta(0,T;L_p(\Omega))$.*

(3) *$\|\mathcal{L}[f]\|_{L_\vartheta(0,T;L_p(\Omega))} \le \|f\|_{L_\vartheta(0,T;L_p(\Omega))}$ (the equality holds if $p<\infty$).*

(4) *If $\boldsymbol{v} \in C^1([0,T];C^1_0(\Omega) \cap J(\Omega))$ and mapping $f: Q_T \to \mathbb{R}$, $f \in C^1(Q_T)$, is represented in Eulerian coordinates then the identity $\mathcal{L}[f](\boldsymbol{\xi},t) = [f]_\xi(\boldsymbol{\xi},t)$ holds, where $[f]_\xi$ is a representation of f in Lagrangian coordinates.*

(5) *There exists an operator $\mathcal{L}^{-1}$ which is inverse to Lagrange operator $\mathcal{L}$. That is, $\mathcal{L} \circ \mathcal{L}^{-1}$ and $\mathcal{L}^{-1} \circ \mathcal{L}$ coincide with the identity mapping in $L_p(\Omega)$ for a. e. $t \in [0,T]$. For $\mathcal{L}$ replaced by $\mathcal{L}^{-1}$ the assertions 1–3 hold true.*

(6) *If $\boldsymbol{v} \in C^1([0,T];C^1_0(\Omega) \cap J(\Omega))$ and function $[f]_\xi: Q_T \to \mathbb{R}$, $[f]_\xi \in C^1(Q_T)$ is represented in Lagrange coordinates then the identity $\mathcal{L}^{-1}[[f]_\xi](\boldsymbol{x},t) = f(\boldsymbol{x},t)$ holds, where f is a representation in Euler coordinates.*

We remark that with the strength of assertions 4 and 6, the notion of Lagrangian transforms is consistent with the classical concept of Lagrange coordinates.

In §3, using proposition 1.8, we will prove the following theorem.

THEOREM 1.9 (On the Lagrange representation of the Tartar equation). *Let U be a 2×2 matrix consisting of the entries $U_{ij}(\boldsymbol{x},t) = \mathcal{L}[\partial_i v_j](\boldsymbol{x},t)$, $i,j=1,2$.*

A non-negative Borel measure μ_t satisfying the condition $d\mu_t(\boldsymbol{x},y) = d\boldsymbol{x}d\nu_{t,x}(y)$, $\nu \in L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))$, solves the Cauchy problem for the Tartar equation (1.5)*, with initial data $\mu_t|_{t=0} = \mu_0$, such that*

$$d\mu_0(\boldsymbol{x},y) = d\boldsymbol{x}d\nu_{0,x}(y), \quad \nu_0 \in L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)),$$

if and only if a non-negative Borel measure η_t, satisfying $d\eta_t(\boldsymbol{x},y) = d\boldsymbol{x}d\lambda_{t,x}(y)$, $\lambda \in L_2(0,T;L_{2,\mathrm{w}}(\Omega;\mathcal{M}_+(S^1)))$, solves the Cauchy problem

$$\partial_t\eta_t + \partial_y(\eta_t U : Y) = 0, \quad \eta_t|_{t=0} = \nu_0. \tag{1.7}$$

Moreover, the relation $\langle\lambda_{t,x},\psi\rangle = \mathcal{L}\left[\langle\nu,\psi\rangle\right](\boldsymbol{x},t)$ holds for any function $\psi\in C(S^1)$ for a.e. $(\boldsymbol{x},t)\in Q_T$.

Finally, we will prove in §4 the well posedness of the Cauchy problem (1.7). Thus, thanks to theorem 1.9, we will establish the validity of theorem 1.4.

2 Lagrange transforms

In this section, unlike the rest of the paper, we consider that a space domain Ω may be not only in $\mathbb{R}^2$, but also in an arbitrary Euclidean space $\mathbb{R}^N$.

2.1 Preliminaries

We will repeatedly use the well-known properties of solutions of the Cauchy problem

$$\partial_t u + \boldsymbol{v}\cdot\nabla_x u = 0, \quad u|_{t=\lambda} = u_0, \quad \boldsymbol{x}\in\Omega, \quad t,\lambda\in[0,T], \tag{2.1}$$

in order to prove proposition 1.8, along with a set of auxiliary statements concerning Lagrange transforms, which will be employed for the verification of theorem 1.9. Therefore, it is suitable to recall some of these properties.

If $\boldsymbol{v}$ and u_0 involved in (2.1) satisfy the conditions $\boldsymbol{v}\in C^1([0,T];C^1_0(\Omega)\cap J(\Omega))$, $u_0\in C^1(\Omega)$, then (2.1) has a unique *classical* solution $u\in C^1(Q_T)$, and this solution has the form [23, §§4–5], [13]

$$u(\boldsymbol{x},t) = u_0(\mathbb{U}^{t,\lambda}(\boldsymbol{x})), \tag{2.2}$$

where $\mathbb{U}^{t_1,t_2} : \Omega\to\Omega$ is a *shift operator*, defined by the identity $\mathbb{U}^{t_1,t_2}(\boldsymbol{x}) = \boldsymbol{\xi}(s)|_{s=t_2}$. Here, $(d/ds)\boldsymbol{\xi} = \boldsymbol{v}(\boldsymbol{\xi},s)$, $\boldsymbol{\xi}|_{s=t_1} = \boldsymbol{x}$. The mapping $\mathbb{U}$ preserves a volume in the sense of the equality $\det(\partial\mathbb{U}^{t_1,t_2}(\boldsymbol{x})/\partial\boldsymbol{x}) = 1$ $\forall\, t_1,t_2\in[0,T]$ [28, ch.II, §5, formula (II.5-8)], and admits the group property $\mathbb{U}^{t_3,t_1}\circ\mathbb{U}^{t_2,t_3} = \mathbb{U}^{t_1,t_2}$ $\forall\, t_1,t_2,t_3\in[0,T]$.

LEMMA 2.1 (Solutions of (2.1) in Lebesgue classes). *Let*

$$\boldsymbol{v} \in L_\gamma(0,T; \overset{\circ}{W}{}^1_\alpha(\Omega) \cap J(\Omega)), \quad u_0 \in L_p(\Omega), \tag{2.3}$$

where $1 \leq \gamma < \infty$, $p^{-1} \leq 1 - (N-\alpha)(\alpha N)^{-1}$ *(in the case of* $N > \alpha$*) or* $p \geq 1$ *is arbitrary (in the case of* $N \leq \alpha$*). Then the following statements are valid.*

(i) There exists a unique solution $u \in L_\infty(0,T; L_p(\Omega))$ *of problem* (2.1). *Furthermore, if* $1 \leq p < \infty$, *then* $u \in C([0,T]; L_p(\Omega))$ *[6, Corollaries II.1 and II.2].*

(ii) The equality $\|u(t)\|_{p,\Omega} = \|u_0\|_{p,\Omega}\ \forall\, t \in [0,T]$ *holds true in the case of* $p \in [1,\infty)$ *[1, Ch.III, §2, Lemma 2.1]. The bound* $\|u(t)\|_{\infty,\Omega} \leq \|u_0\|_{\infty,\Omega}\ \forall t \in [0,T]$ *holds true in the case of* $p = \infty$ *[6, Formula (16)].*

(iii) Let $\boldsymbol{v}_n$ *and* u_{0n} *satisfy* (2.3), *with* $p < \infty$, *and* $\boldsymbol{v}_n \to \boldsymbol{v}$ *in* $L_\gamma(0,T; W^1_\alpha(\Omega) \cap J(\Omega))$, $u_{0n} \to u_0$ *in* $L_p(\Omega)$ *as* $n \to \infty$. *Let* $\{u_n\}$ *be a sequence of solutions of* (2.1) *with given functions* $\boldsymbol{v}$ *and* u_0 *replaced by* $\boldsymbol{v}_n$ *and* u_{0n}. *Let* $\{u_n\}$ *be bounded in* $L_\infty(0,T; L_p(\Omega))$. *Then* $u_n \to u$ *in* $C([0,T]; L_p(\Omega))$, *where* u *is a solution of* (2.1) *with given functions* $\boldsymbol{v}$ *and* u_0 *[6, Theorem II.4].*

(iv) Let $u \in L_\infty(0,T; L_p(\Omega))$ *be a solution of problem* (2.1). *Consider the function* $u_\varepsilon = u * \omega_\varepsilon$, *where* $\omega_\varepsilon = \varepsilon^{-1}\omega(\cdot/\varepsilon^N)$, ω *is an even function of class* $\mathcal{D}_+(\mathbb{R}^N)$ *with mean value equal to zero. Then one has* $\partial_t u_\varepsilon + \operatorname{div}_x(\boldsymbol{v} u_\varepsilon) = r_\varepsilon$, *where* $r_\varepsilon \to 0$ *in* $L_\gamma(0,T; L_\beta(\Omega))$, $\beta^{-1} = \alpha^{-1} + p^{-1}$ *in the case of* $\alpha < \infty$ *or* $p < \infty$; $\beta < \infty$ *is arbitrary in the case* $\alpha = p = \infty$ *[6, Theorem II.1].*

We denote $\|\cdot\|_{q,\Omega} = \|\cdot\|_{L_q(\Omega)}\ \forall\, q \in [1,\infty]$.

REMARK 2.1. If a sequence $\{f_n(\boldsymbol{x},t)\}$ is bounded in $L_\infty(Q_T)$ and converges to a function f in $C([0,T]; L_p(\Omega))\ \forall\, p < +\infty$ then $f_n \to f$ in $L_\infty(Q_T)$ weak-star. This evident proposition along with the assertion (iii) in Lemma 2.1 leads to the following.

COROLLARY 2.3. *If hypothesis in the assertion (iii) of lemma 2.1 holds and a sequence* u_{0n} *is bounded in* $L_\infty(\Omega)$ *then* $u_n \to u$ *in* $L_\infty(Q_T)$ *weak*.*

DEFINITION 2.4. A vector function $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x},t,\lambda)$, where X_i, $i = 1,\dots,N$, are solutions of (2.1) provided with Cauchy data $X_i|_{t=\lambda} = x_i$, is called *a flow*.

Definition 2.4, the representation (2.2), assertions (ii) and (iv) of lemma 2.1 and corollary 2.3 directly imply the following lemma.

LEMMA 2.5 (On properties of a flow). *If a velocity field* $\boldsymbol{v}$ *in* (2.1) *satisfies* (2.3), *then we have the following.*

(i) $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t, \lambda) \in \overline{\Omega}\ \forall\, t, \lambda \in [0, T]$ *and for a. e.* $\boldsymbol{x} \in \Omega$*; [4, §1.2]*

(ii) if $f \in C^1(\overline{\Omega})$ *and* $\boldsymbol{X}_\varepsilon$ *is the regularization of a flow in the sense of assertion (iv) of lemma 2.1 then* $f(\boldsymbol{X}_\varepsilon) \to f(\boldsymbol{X})$ *in* $L_{\vartheta_1}(0, T; L_{\vartheta_2}(\Omega))\ \forall\, \vartheta_1,\, \vartheta_2 < \infty$ *and in* $L_\infty(Q_T)$ *weak*.*

(iii) $\partial_t f(\boldsymbol{X}_\varepsilon) + \boldsymbol{v} \cdot \nabla_x f(\boldsymbol{X}_\varepsilon) \to 0$ *in* $L_\gamma(0, T; L_\beta(\Omega))$*, where* β *is defined in assertion (iv) of lemma 2.1.*

We denote by $\overline{\Omega}$ a closed set $\Omega \cup \partial\Omega$.

2.2 Properties of Lagrange operator

2.2.1 Proof of proposition 1.8

Verification of the assertions 2–6 are simple, so, let us confine ourselves to the very scheme of the proof. In the case of smooth $\boldsymbol{v}$ and f the assertion 2 clearly appears from the properties of a classical solutions of (2.1). Assertion (iii) of lemma 2.1 makes it possible to extend this fact onto the case of nonsmooth $\boldsymbol{v}$ and f. The bounds in assertion 3 are derived from assertion (ii) of lemma 2.1. The correctness of assertion 4 is based on the representation (2.2) for a classical solution of (2.1), which coincides with the representation of a function f in Lagrange coordinates $\boldsymbol{\xi}$. The inverse operator $\mathcal{L}^{-1}$ can be introduced for a.e. $t \in [0, T]$ by means of a solution of the Cauchy problem

$$\partial_s R^{(t)} + \boldsymbol{v}(\boldsymbol{x}, s) \cdot \nabla_x R^{(t)} = 0,\ (\boldsymbol{x}, s) \in Q_T, \quad R^{(t)}(\boldsymbol{x}, s)|_{s=0} = f(\boldsymbol{x}, t),\ \boldsymbol{x} \in \Omega,$$

using the formula $\mathcal{L}^{-1}[f](\boldsymbol{x}, t) = R^{(t)}(\boldsymbol{x}, t)$. Thus validity of the assertions 5 and 6 for smooth $\boldsymbol{v}$ and f is evident due to the representation (2.2), and can be extended on the case of nonsmooth $\boldsymbol{v}$ and f due to assertion (iii) of lemma 2.1.

Now, let us give a detailed justification of assertion 1, concerning the measurability of $\mathcal{L}[f]$. Notice that $L_\infty(\Omega) \subset L_r(\Omega)\ \forall\, r < \infty$, since Ω is a bounded domain. Therefore, it is enough to confine ourselves to the case $p < \infty$, $\alpha < \infty$. At first, assume f is in $C^1(Q_T)$. Let $\{\boldsymbol{v}_n\} \subset C^1([0, T], C_0^1(\Omega) \cap J(\Omega))$, $\boldsymbol{v}_n \to \boldsymbol{v}$ in $L_\gamma(0, T; \overset{\circ}{W}{}^1_\alpha(\Omega))$ and $\mathcal{L}_n$ is Lagrange operator associated with $\boldsymbol{v}_n$. Definition 1.1 and the properties of a classical solution of (2.1) yield $\mathcal{L}_n[f] \in C^1(Q_T)$. Due to assertion (iii) of lemma 2.1, the relation $\mathcal{L}_n[f](\cdot, t) \underset{n\to\infty}{\longrightarrow} \mathcal{L}[f](\cdot, t)$ in $L_p(\Omega)\ \forall\, t \in [0, T]$ is valid. Hence $\mathcal{L}_n[f] \to \mathcal{L}[f]$ a. e. in Q_T. This means that if $f \in C^1(Q_T)$ then $\mathcal{L}[f]$ is measurable in Q_T.

Now assume $f \in L_\vartheta(0, T; L_p(\Omega))$. Let $f_n \in C^1(Q_T)$, $n = 1, 2, \ldots$, $f_n \to f$ in $L_\vartheta(0, T; L_p(\Omega))$. As it has just been proved, $\mathcal{L}[f_n]$ is measurable in Q_T. As a consequence of definition 1.6, the linearity of (2.1) and the validity of the assertion (i) in lemma 2.1, one has $\mathcal{L}[f_n - f] = \mathcal{L}[f_n] - \mathcal{L}[f]$ for $n \geq 1$. Moreover, due to the equality in the hypothesis in assertion (ii) of lemma 2.1, we arrive at the identity $\|\mathcal{L}[f_n](\cdot, t) - \mathcal{L}[f](\cdot, t)\|_{p,\Omega} = \|f_n(\cdot, t) - f(\cdot, t)\|_{p,\Omega}$ for a. e. $t \in [0, T]$. This yields the limiting relation $\|\mathcal{L}[f_n](\cdot, t) - \mathcal{L}[f](\cdot, t)\|_{p,\Omega} \to 0$ as $n \to \infty$

for a.e. $t \in [0,T]$. Hence, $\mathcal{L}[f_n] \to \mathcal{L}[f]$ a. e. in Q_T. By this, we establish that $\mathcal{L}[f]$ is a measurable function, and thus conclude the proof of proposition 1.8.

2.2.2 Auxiliary properties of Lagrange operator

We need to establish some extra properties of Lagrange operator, which will be necessary for verification of theorem 1.9.

PROPOSITION 2.6.

(1) If $f \in L_\vartheta(0,T;L_1(\Omega))$ then $\int_\Omega \mathcal{L}[f]d\boldsymbol{x} = \int_\Omega f\, d\boldsymbol{x}$ a. e. in $[0,T]$.

(2) If $f_i \in L_{\vartheta_i}(0,T;L_{p_i}(\Omega))$, $i = 1,\dots,k$, where $\sum_{i=1}^k p_i^{-1} \le 1$, $\sum_{i=1}^k \vartheta_i^{-1} \le 1$, then $\mathcal{L}[f_1 \dots f_k] \in L_1(Q_T)$ and $\mathcal{L}[f_1 \dots f_k](\boldsymbol{x},t) = \mathcal{L}[f_1](\boldsymbol{x},t) \dots \mathcal{L}[f_k](\boldsymbol{x},t)$ a. e. in Q_T.

(3) If $f_i \in L_1(Q_T)$, $i = 1,\dots,k$, then $\mathcal{L}[f_1 + \dots + f_k] \in L_1(Q_T)$ and

$$\mathcal{L}[f_1 + \dots + f_k](\boldsymbol{x},t) = \mathcal{L}[f_1](\boldsymbol{x},t) + \dots + \mathcal{L}[f_k](\boldsymbol{x},t) \text{ a. e. in } Q_T.$$

(4) If $f \in C^1(Q_T)$, $\boldsymbol{X}^0(\boldsymbol{x},t) = \boldsymbol{X}(\boldsymbol{x},t,0)$ is a flow in the sense of the definition 2.4, then $\mathcal{L}[f \circ \boldsymbol{X}^0](\boldsymbol{x},t) = f(\boldsymbol{x},t)$ a. e. in Q_T, where $[f \circ \boldsymbol{X}^0](\boldsymbol{x},t) = f(\boldsymbol{X}^0(\boldsymbol{x},t),t)$.

(5) If $f \in C[0,T]$ then $\mathcal{L}[f](\boldsymbol{x},t) = f(t)\ \forall\, (\boldsymbol{x},t) \in Q_T$.

(6) If $f(\boldsymbol{x},t) \in L_\vartheta(0,T;L_p(\Omega))$ and $f(\boldsymbol{x},t) \ge 0$ then $\mathcal{L}[f](\boldsymbol{x},t) \ge 0$.

(7) If $\boldsymbol{v}_n$, $\boldsymbol{v} \in L_\gamma(0,T;W^1_\alpha(\Omega) \cap J(\Omega))$, $\boldsymbol{v}_n \to \boldsymbol{v}$ in $L_\gamma(0,T;W^1_{\alpha_1}(\Omega))$, where $\alpha_1 < \infty$, $\alpha_1 \le \alpha$; $\mathcal{L}_n$, $\mathcal{L}$ are Lagrange operators associated with $\boldsymbol{v}_n$, $\boldsymbol{v}$, respectively, and $f \in L_\vartheta(0,T;L_p(\Omega))$, where $1 < \vartheta, p$, then $\mathcal{L}_n[f] \to \mathcal{L}[f]$ in $L_\vartheta(0,T;L_p(\Omega))$ weak.*

(8) Assertions 1–3 and 5–7 hold true with $\mathcal{L}$ replaced by $\mathcal{L}^{-1}$.

Proof. For the sake of brevity, we omit proofs of the assertions 1–6, 8, because the methods of proofs do not differ from those of the proof given for the assertions 2–6 of proposition 1.8. That is, we verify the assertions in the case of smooth $\boldsymbol{v}$ and f, and apply assertion (iii) of lemmas 2.1 and 2.5 and proposition 1.8 in order to give an extension on the case of nonsmooth $\boldsymbol{v}$ and f.

Now we give justification of assertion 7 in detail. On the strength of Banach–Steinhaus theorem [12, ch.VII, §1, theorem 3], it is enough to prove that $\sup\limits_n \|\mathcal{L}_n[f]\|_{L_\vartheta(0,T;L_p(\Omega))} < \infty$ and

$$\int_{Q_T} V(\boldsymbol{x},t)\mathcal{L}_n[f](\boldsymbol{x},t)d\boldsymbol{x}dt \to \int_{Q_T} V(\boldsymbol{x},t)\mathcal{L}[f](\boldsymbol{x},t)d\boldsymbol{x}dt \tag{2.4}$$

for any V in some set being dense in the space $L_{\vartheta'}(0,T;L_{p'}(\Omega))$, $p^{-1}+(p')^{-1}=1$, $\vartheta^{-1}+(\vartheta')^{-1}=1$.

The estimates in assertion 3 of proposition 1.8 yield that $\mathcal{L}_n[f]$ is bounded in $L_\vartheta(0,T;L_p(\Omega))$. Let us construct a dense in $L_{\vartheta'}(0,T;L_{p'}(\Omega))$ set of functions satisfying (2.4). Thus, we will conclude the proof of the assertion 6. The assertion (iii) in Lemma 2.1 implies $\|\mathcal{L}_n[f](t)-\mathcal{L}[f](t)\|_{p,\Omega}\to 0$ a. e. in $t\in[0,T]$. Hence, $\mathcal{L}_n[f]\to\mathcal{L}[f]$ a. e. in Q_T. This fact along with Egorov theorem yields that for any $\varepsilon>0$ there exists a set $\overline{Q_T^\varepsilon}\subset Q_T$ such that $\mathtt{meas}\,Q_T-\mathtt{meas}\,Q_T^\varepsilon<\varepsilon$ and $\mathcal{L}_n[f](\boldsymbol{x},t)\to\mathcal{L}[f](\boldsymbol{x},t)$ uniformly in $\overline{Q_T^\varepsilon}$. Hence, $\int_E\mathcal{L}_n[f]d\boldsymbol{x}dt\to\int_E\mathcal{L}[f]d\boldsymbol{x}dt$, where E is an arbitrary measurable subset of $\overline{Q_T^\varepsilon}$. Consider the sequence of numbers $\varepsilon_k\to 0$ and the sequence of domains $Q_T^{\varepsilon_k}$ correspondent to it in the sense pointed in the previous sentence. Consider the set consisting of characteristic functions of all measurable subsets $E(\varepsilon_k)$ in $Q_T^{\varepsilon_k}$, $k=1,2,\ldots$. At the end, it remains to note that the linear span of this set is dense in $L_{\vartheta'}(0,T;L_{p'}(\Omega))$ [12, ch.III, §3, theorem 4, corollary 2]. □

3 Lagrange representation of Tartar equation

3.1 Definitions of measure-valued solutions of Cauchy problems for Tartar equation and Equation (1.7)

Nonnegative measure-valued solutions of Cauchy problems for Tartar equation (1.5) and Equation (1.7) are understood in the sense of the following definitions.

Definition 3.1. By a *nonnegative measure-valued solution* of Cauchy problem for Equation (1.5) we mean a measure μ_t such that $d\mu_t(\boldsymbol{x},y)=d\boldsymbol{x}\,d\nu_{t,x}(y)$, where $\nu\in L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))$, and for all $\tau\in[0,T]$ the equality

$$\int_0^\tau dt\int_{\Omega\times S^1}(\partial_t\Phi+\boldsymbol{v}\cdot\nabla_x\Phi+(Y\ :\nabla_x\boldsymbol{v})\,\partial_y\Phi)d\mu_t(\boldsymbol{x},y)$$
$$=\int_{\Omega\times S^1}\Phi(\boldsymbol{x},y,\tau)d\mu_\tau(\boldsymbol{x},y)-\int_{\Omega\times S^1}\Phi(\boldsymbol{x},y,0)d\mu_0(\boldsymbol{x},y)\quad(3.1)$$

takes place. Here, $\Phi\in C^1([0,T]\times\Omega\times S^1)$ is a test function satisfying $\Phi|_{\partial\Omega}=0$. •

Definition 3.2. By a *nonnegative measure-valued solution* of Cauchy problem for Equation (1.7) we mean a measure η_t such that $\eta_t(\boldsymbol{x},y)=d\boldsymbol{x}d\lambda_{t,x}(y)$, where $\lambda\in L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))$, and for all $\tau\in[0,T]$ the equality

$$\int_0^\tau dt\int_\Omega d\boldsymbol{x}\int_{S^1}\Big(\partial_t\Phi+(U(\boldsymbol{x},t)\ :\ Y)\partial_y\Phi\Big)d\lambda_{t,x}(y)$$
$$=\int_\Omega d\boldsymbol{x}\int_{S^1}\Phi(\boldsymbol{x},y,\tau)d\lambda_{\tau,x}(y)-\int_\Omega d\boldsymbol{x}\int_{S^1}\Phi(\boldsymbol{x},y,0)d\lambda^0_x(y)\quad(3.2)$$

takes place for any test function $\Phi\in L_\infty(\Omega,C^1(S^1\times[0,T]))$. •

Remark that the first integral in the right hand side of (3.1) makes sense for any $\tau\in[0,T]$, because whenever μ_t satisfies (1.5) in the sense of the theory of distributions, $t\to\mu_t$ is a weakly continuous mapping of interval $[0, T]$ into the space of Borel measures $\Omega \times S^1$. Verification of this is trivial and can be fulfilled similarly to, for example, [1, Ch. III, §1]. The same observation is true for (3.2) as well.

3.2 Proof of Theorem 1.9

At first, we prove that if μ_t is a nonnegative measure-valued solution of Cauchy problem for Equation (1.5) then measure η_t defined in the formulation of Theorem 1.9 is a nonnegative measure-valued solution of Cauchy problem for Equation (1.7).

Justification of this is based on a special choice of test functions in (3.1). Let $\varphi_1 \in C_0^1(\Omega)$, $\varphi_2 \in C^1[0, T]$, $\varphi_3 \in C^1(S^1)$. Assume $\varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, t) = ((\varphi_1 \circ (\boldsymbol{X}^0 * \omega_\varepsilon)) * \bar{\omega}_\sigma)(\boldsymbol{x}, t)$, where $\boldsymbol{X}^0(\boldsymbol{x}, t) = \boldsymbol{X}(\boldsymbol{x}, t, 0)$ is a flow in the sense of Definition 2.1, $\omega_\varepsilon(\boldsymbol{x})$ is a regularizing kernel defined in Lemma 2.1, $\bar{\omega}_\sigma(t) = \sigma^{-1}\bar{\omega}(t\sigma^{-1})$ is a kernel mollifying with respect to t, such that $\bar{\omega}$ is an even function in the class $\mathcal{D}_+(\mathbb{R})$ with the mean value equal to one. Introducing a test function of a form $\Phi_\varepsilon(\boldsymbol{x}, y, t) = \varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, t)\varphi_2(t)\varphi_3(y)$ into the equality (3.1) we obtain

$$\begin{aligned}
&\int_0^\tau dt \int_{\Omega\times S^1} \Big(\partial_t \varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, t)\varphi_3(y) + \boldsymbol{v} \cdot \nabla_x \varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, t)\varphi_3(y) \\
&\qquad\qquad + (Y \ : \nabla_x \boldsymbol{v})\ \varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, t)\partial_y \varphi_3(y) \Big) \varphi_2(t) d\mu_t(\boldsymbol{x}, y) \\
&\qquad\qquad + \int_0^\tau dt \int_{\Omega\times S^1} \varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, t)\partial_t\varphi_2(t)\varphi_3(y) d\mu_t(\boldsymbol{x}, y) \\
&= \int_{\Omega\times S^1} \varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, \tau)\varphi_2(\tau)\varphi_3(y) d\mu_\tau(\boldsymbol{x}, y) - \int_{\Omega\times S^1} \varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, 0)\varphi_2(0)\varphi_3(y) d\mu_0(\boldsymbol{x}, y). \quad (3.3)
\end{aligned}$$

Due to Lemma ?? the set $\{\varphi_1^{\sigma\varepsilon}(\boldsymbol{x}, t)\}_{\varepsilon,\sigma>0}$ is uniformly bounded in $L_\infty(Q_T)$, and the following relations hold true.

$$\varphi_1^{\sigma\varepsilon} \to \varphi_1 \circ \boldsymbol{X}^0 \text{ in } L_\vartheta(Q_T), \text{ and weak-star in } L_\infty(Q_T), \quad (3.4)$$

$$\varphi_1^{\sigma\varepsilon}(\tau) \to (\varphi_1 \circ \boldsymbol{X}^0)(\tau),\ \varphi_1^{\sigma\varepsilon}(0) \to (\varphi_1 \circ \boldsymbol{X}^0)(0) \text{ in } L_\vartheta(\Omega), \text{ and weak-star in } L_\infty(\Omega), \quad (3.5)$$

$$\partial_t \varphi_1^{\sigma\varepsilon} + \boldsymbol{v} \cdot \nabla_x \varphi_1^{\sigma\varepsilon} \to 0 \text{ in } L_2(Q_T) \quad (3.6)$$

as $\varepsilon, \sigma \to 0$, where $\vartheta < \infty$ is arbitrary, $(\varphi_1 \circ \boldsymbol{X}^0)(0) = \varphi_1(\boldsymbol{x})$ is true in view of the definition of the flow. Passing to the limit in (3.3) and taking into account the hypothesis in Theorem 1.3 and relations (3.4)–(3.6) we arrive at the equality

$$\begin{aligned}
&\int_0^\tau dt \int_{\Omega\times S^1} \Big((\varphi_1 \circ \boldsymbol{X}^0)(\boldsymbol{x}, t)\partial_t\varphi_2(t)\varphi_3(y) \\
&\qquad\qquad + (Y \ : \nabla_x \boldsymbol{v})\ (\varphi_1 \circ \boldsymbol{X}^0)(\boldsymbol{x}, t)\varphi_2(t)\partial_y\varphi_3(y) \Big) d\mu_t(\boldsymbol{x}, y) \\
&= \int_{\Omega\times S^1} (\varphi_1 \circ \boldsymbol{X}^0)(\boldsymbol{x}, \tau)\varphi_2(\tau)\varphi_3(y) d\mu_\tau(\boldsymbol{x}, y) - \int_{\Omega\times S^1} \varphi_1(\boldsymbol{x})\varphi_2(0)\varphi_3(y) d\mu_0(\boldsymbol{x}, y). \quad (3.7)
\end{aligned}$$

Using the representation (1.6) for measure μ_t this equality can be rewritten in the form

$$\int_0^\tau dt \int_\Omega \Big((\varphi_1 \circ \boldsymbol{X}^0)(\boldsymbol{x},t)\partial_t \varphi_2(t) \int_{S^1} \varphi_3(y) d\nu_{t,x}(y) + \sum_{i,j=1}^{2} (\varphi_1 \circ \boldsymbol{X}^0)(\boldsymbol{x},t)\varphi_2(t)\partial_i v_j(\boldsymbol{x},t) \int_{S^1} Y_{ij}\, \partial_y \varphi_3(y) d\nu_{t,x}(y) \Big) d\boldsymbol{x}$$

$$= \int_\Omega d\boldsymbol{x} (\varphi_1 \circ \boldsymbol{X}^0)(\boldsymbol{x},\tau)\varphi_2(\tau) \int_{S^1} \varphi_3(y) d\nu_{\tau,x}(y) - \int_\Omega d\boldsymbol{x}\, \varphi_1(\boldsymbol{x})\varphi_2(0) \int_{S^1} \varphi_3(y) d\nu_{0,x}(y). \quad (3.8)$$

Now there arises a question of extending the concept of Lagrange transform in order to treat measures. An answer is given in the following lemma.

LEMMA 3.3 (On Lagrange representation of measures). *If*

$$\nu \in L_2(0,T; L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1)))$$

then there exists a unique measure $\lambda \in L_2(0,T; L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1)))$ *satisfying the identities*

$$\langle \lambda_{t,x}, \psi \rangle = \mathcal{L}[\langle \nu, \psi \rangle](\boldsymbol{x},t) \quad \forall \psi \in C(S^1), \text{ for a. e. } (\boldsymbol{x},t) \in Q_T; \quad (3.9)$$

$$\|\lambda_{t,x}\| = \mathcal{L}[\|\nu\|](\boldsymbol{x},t) \text{ for a. e. } (\boldsymbol{x},t) \in Q_T; \quad (3.10)$$

and the bound

$$\|\langle \lambda, \psi \rangle\|_{1,Q_T} \le \|\nu\|_{L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))} \|\psi\|_{L_2(Q_T,C(S^1))} \quad \forall \psi \in L_2(Q_T, C(S^1)). \quad (3.11)$$

This statement is also true with $\mathcal{L}$ *replaced by* $\mathcal{L}^{-1}$.

Proof. Proposition 2.6 yields that $\mathcal{L}[\langle \nu, \cdot \rangle](\boldsymbol{x},t)$ is a linear functional defined on $C(S^1)$ for a. e. $(\boldsymbol{x},t) \in Q_T$. Indeed, due to the assertion 3 in Propositon 2.6 for all ψ_1, $\psi_2 \in C(S^1)$ the chain of equalities

$$\mathcal{L}[\langle \nu, \psi_1 + \psi_2 \rangle](\boldsymbol{x},t) = \mathcal{L}[\langle \nu, \psi_1 \rangle + \langle \nu, \psi_2 \rangle](\boldsymbol{x},t) = \mathcal{L}[\langle \nu, \psi_1 \rangle](\boldsymbol{x},t) + \mathcal{L}[\langle \nu, \psi_2 \rangle](\boldsymbol{x},t) \quad (3.12)$$

holds for a. e. $(\boldsymbol{x},t) \in Q_T$, and due to the assertions 2 and 5 for any function $\psi(y) \in C(S^1)$ and any constant c the identities

$$\mathcal{L}[\langle \nu, c\psi \rangle](\boldsymbol{x},t) = \mathcal{L}[c\, \langle \nu, \psi \rangle](\boldsymbol{x},t) = \mathcal{L}[c](\boldsymbol{x},t)\, \mathcal{L}[\langle \nu, \psi \rangle](\boldsymbol{x},t) = c\mathcal{L}[\langle \nu, \psi \rangle](\boldsymbol{x},t) \quad (3.13)$$

are valid a. e. in Q_T. Since measure $\nu_{t,x}$ is nonnegative, the assertion 6 in Proposition 2.6 yields the inequality

$$\mathcal{L}[\langle \nu, \psi \rangle](\boldsymbol{x},t) \ge 0 \text{ for a. e. } (\boldsymbol{x},t) \in Q_T \quad (3.14)$$

for all nonnegative in S^1 functions $\psi \in C(S^1)$. Now suppose that $\psi \in C(S^1)$ is an arbitrary (not necessarily nonnegative) function. In the strength of (3.14) the inequalities

$\mathcal{L}[\langle\nu, \|\psi\|_{C(S^1)}\rangle](\boldsymbol{x},t) \geq \mathcal{L}[\langle\nu,\psi\rangle](\boldsymbol{x},t)$ and $\mathcal{L}[\langle\nu,\psi\rangle](\boldsymbol{x},t) \geq -\mathcal{L}[\langle\nu, \|\psi\|_{C(S^1)}\rangle](\boldsymbol{x},t)$ are valid for a. e. $(\boldsymbol{x},t)\in Q_T$. Due to the assertion 6 in Proposition 2.6 we obtain

$$\mathcal{L}[\langle\nu,\psi\rangle](\boldsymbol{x},t) \leq \|\psi\|_{C(S^1)}\mathcal{L}[\langle\nu,1\rangle](\boldsymbol{x},t) \text{ for a. e. } (\boldsymbol{x},t)\in Q_T. \tag{3.15}$$

The relations (3.12)–(3.15) yield that for a. e. $(\boldsymbol{x},t)\in Q_T$ the functional $(\boldsymbol{x},t)\to\mathcal{L}[\langle\nu,\cdot\rangle](\boldsymbol{x},t)$ defines a nonnegative Borel measure $\lambda_{t,x}$ on S^1. Thus, the formula (3.9) makes sense. Uniqueness of measure $\lambda_{t,x}$ follows from the representation (3.9) along with the identity $\mathcal{L}[0](\boldsymbol{x},t)\equiv 0$. Since

$$\langle\lambda_{t,x},1\rangle = \mathcal{L}[\langle\nu,1\rangle](\boldsymbol{x},t) = \mathcal{L}[\|\nu\|](\boldsymbol{x},t) \text{ for a. e. } (\boldsymbol{x},t)\in Q_T$$

the formula (3.10) is correct. Next, if $\psi\in L_2(Q_T, C(S^1))$ then

$$\|\langle\lambda,\psi\rangle\|_{1,Q_T} \leq \|\max_{y\in S^1}|\psi|\,\langle\lambda,1\rangle\|_{1,Q_T} = \big\|\max_{y\in S^1}|\psi|\,\|\lambda\|\big\|_{1,Q_T}.$$

Estimating this expression by means of Cauchy–Schwartz–Bunyakovskii inequality we arrive at the bound

$$\|\langle\lambda,\psi\rangle\|_{1,Q_T} \leq \|\psi\|_{L_2(Q_T,C(S^1))}\|\lambda\|_{L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))}. \tag{3.16}$$

In completion of the proof of the lemma, notice that the equality $\|\lambda\|_{L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))} = \|\nu\|_{L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))}$ holds true due to (3.10) and the assertion 3 in Proposition 1.8. Hence, in view of (3.16), the formula (3.11) is valid. ▲

Turn back to verification of Theorem 1.9. Consider Lagrange transforms of the functions involved in (3.8). Denote

$$\langle\lambda_{t,x},\varphi_3\rangle = \mathcal{L}[\langle\nu,\varphi_3\rangle](\boldsymbol{x},t), \tag{3.17}$$

$$\langle\lambda_{t,x}, Y_{ij}\partial_y\varphi_3\rangle = \mathcal{L}[\langle\nu, Y_{ij}\partial_y\varphi_3\rangle](\boldsymbol{x},t),\ \ i,j=1,2, \tag{3.18}$$

$$U_{ij}(\boldsymbol{x},t) = \mathcal{L}[\partial_i v_j](\boldsymbol{x},t),\ \ i,j=1,2. \tag{3.19}$$

Observe that $U_{ij}\in L_2(Q_T)$, $i,j=1,2$, due to the assertion 2 in Proposition 1.8, and $\lambda\in L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))$ due to Lemma 3.3. The assertion 4 in Proposition 2.6 yields

$$\mathcal{L}[\varphi_1\circ\boldsymbol{X}^0](\boldsymbol{x},t) = \varphi_1(\boldsymbol{x}),\ \ \boldsymbol{x}\in\Omega. \tag{3.20}$$

Using (3.17)–(3.20) and basing on Proposition 2.6 we rewrite (3.8) in the equivalent form

$$\begin{aligned}\int_0^\tau dt\int_\Omega\Bigg(&\varphi_1(\boldsymbol{x})\partial_t\varphi_2(t)\int_{S^1}\varphi_3(y)d\lambda_{t,x}(y)\\ &+\sum_{i,j=1}^2\varphi_1(\boldsymbol{x})\varphi_2(t)U_{ij}(\boldsymbol{x},t)\int_{S^1}Y_{ij}\,\partial_y\varphi_3(y)d\lambda_{t,x}(y)\Bigg)d\boldsymbol{x}\\ =\int_\Omega d\boldsymbol{x}\varphi_1(\boldsymbol{x})\varphi_2(\tau)&\int_{S^1}\varphi_3(y)d\lambda_{\tau,x}(y) - \int_\Omega d\boldsymbol{x}\varphi_1(\boldsymbol{x})\varphi_2(0)\int_{S^1}\varphi_3(y)d\lambda_{0,x}(y).\end{aligned} \tag{3.21}$$

Here, in line with Definition 1.4 of Lagrange transform we have $\lambda_{0,x}(y) = \nu_{0,x}(y)$. Also, in view of Definition 3.2 let us denote by η_t a measure defined on $\Omega \times S^1$ by virtue of the decomposition $d\eta_t(\boldsymbol{x}, y) = d\boldsymbol{x} d\lambda_{t,x}(y)$.

Any function $\Phi \in L_\infty(\Omega, C^1(S^1 \times [0,T]))$ can be approximated by a sequence of functions Φ_n from the linear span of the set $\{\varphi_1(\boldsymbol{x})\varphi_2(t)\varphi_3(y) \,|\, \varphi_1 \in C_0^1(\Omega),\ \varphi_2 \in C^1[0,T],\ \varphi_3 \in C^1(S^1)\}$ in the way that $\Phi_n \to \Phi$ weak-star in $L_\infty(\Omega, C^1(S^1 \times [0,T]))$. In the strength of this limiting relation, Lemma 3.3, and the hypothesis in Theorem 1.9 the integral equality (3.2) follows from (3.21).

By this we have proved the straightforward assertion in Theorem 1.9. That is, we established that a $\mathcal{L}$-image of a nonnegative measure-valued solution of Cauchy problem for Tartar equation is a nonnegative measure-valued solution of Cauchy problem for Equation (1.7).

Now, we will prove the inverse assertion in the theorem. That is, we will prove that $\mathcal{L}^{-1}$-image of a solution of Cauchy problem for Equation (1.7) solves Cauchy problem for Equation (1.5).

The proof is based on the special choice of test functions in the integral equality (3.2).

Assume $\{\boldsymbol{v}_n\} \subset C^1([0,T], C_0^1(\Omega) \cap \dot{J}(\Omega))$, $\boldsymbol{v}_n \to \boldsymbol{v}$ in $L_2(0,T; \dot{J}^1(\Omega))$, where $\boldsymbol{v}$ is a vector-field associated with Lagrange transform $\mathcal{L}$. Denote by $\mathbb{U}_n^{0,t}$ the shift operator associated with $\boldsymbol{v}_n$. Let $\varphi_1 \in C^1[0,T]$, $\varphi_2 \in C^1(S^1)$, $\varphi_3 \in C_0^1(\Omega)$. Introducing into (3.2) a test function of a form $\Phi(t, \boldsymbol{x}, y) = \varphi_1(t)\varphi_2(y)\varphi_3(\mathbb{U}_n^{0,t}(\boldsymbol{x}))$ and observing that due to the definition of shift operators stated in §2.1 the formula

$$\frac{d\mathbb{U}_n^{0,t}(\boldsymbol{x})}{dt} = \boldsymbol{v}_n(\mathbb{U}_n^{0,t}(\boldsymbol{x}), t) \tag{3.22}$$

holds true, we arrive at the equality

$$\begin{aligned} &\int_0^\tau dt \int_\Omega d\boldsymbol{x} \Bigg[\Bigg(\partial_t \varphi_1(t) \varphi_3(\mathbb{U}_n^{0,t}(\boldsymbol{x})) + \varphi_1(t) \Big(\frac{d\mathbb{U}_n^{0,t}(\boldsymbol{x})}{dt} \cdot (\nabla_x \varphi_3 \circ \mathbb{U}_n^{0,t})(\boldsymbol{x}) \Big) \Bigg) \int_{S^1} \varphi_2(y) d\lambda_{t,x}(y) \\ &\qquad + \sum_{i,j=1}^{2} \varphi_1(t) \varphi_3(\mathbb{U}_n^{0,t}(\boldsymbol{x})) U_{ij}(\boldsymbol{x}, t) \int_{S^1} Y_{ij}(y) \partial_y \varphi_2(y) d\lambda_{t,x}(y) \Bigg] \\ &= \int_\Omega d\boldsymbol{x}\, \varphi_1(\tau) \varphi_3(\mathbb{U}_n^{0,\tau}(\boldsymbol{x})) \int_{S^1} \varphi_2(y) d\lambda_{\tau,x}(y) - \int_\Omega d\boldsymbol{x}\, \varphi_1(0) \varphi_3(\boldsymbol{x}) \int_{S^1} \varphi_2(y) d\lambda_{0,x}(y). \end{aligned} \tag{3.23}$$

In the strength of the assertion 4 in Proposition 1.8 we have $\varphi_3(\mathbb{U}_n^{0,t}(\boldsymbol{x})) = \mathcal{L}_n[\varphi_3](\boldsymbol{x}, t)$ and $(\nabla_x \varphi_3 \circ \mathbb{U}_n^{0,t})(\boldsymbol{x}) = \mathcal{L}_n[\nabla_x \varphi_3](\boldsymbol{x}, t)$. Also, in view of (3.22) $(d/dt)\mathbb{U}_n^{0,t}(\boldsymbol{x}) = \mathcal{L}_n[\boldsymbol{v}_n](\boldsymbol{x}, t)$ takes place.

Let us introduce these expressions into the equality (3.23) and pass to the limit as $n \to \infty$. In the strength of the assertion 3 in Proposition 1.8 the equalities

$$\mathcal{L}_n[\boldsymbol{v}_n](\boldsymbol{x}, t) \mathcal{L}_n[\nabla_x \varphi_3](\boldsymbol{x}, t) = \mathcal{L}_n[\boldsymbol{v}_n - \boldsymbol{v}](\boldsymbol{x}, t) \mathcal{L}_n[\nabla_x \varphi_3](\boldsymbol{x}, t) + \mathcal{L}_n[\boldsymbol{v}](\boldsymbol{x}, t) \mathcal{L}_n[\nabla_x \varphi_3](\boldsymbol{x}, t)$$

are valid a. e. in Q_T. As a consequence of the assertion 3 in Proposition 1.8 the identities $\|\mathcal{L}_n[\partial_j v_{ni} - \partial_j v_i]\|_{2,Q_T} = \|\partial_j v_{ni} - \partial_j v_i\|_{2,Q_T}$, $i,j = 1,2$, take place. Hence, $\mathcal{L}_n[\boldsymbol{v}_n - \boldsymbol{v}] \to 0$

in $L_2(0,T;W_2^1(\Omega))$. It follows from the assertion 7 in Proposition 2.6 that $\mathcal{L}_n[\varphi_3] \to \mathcal{L}[\varphi_3]$ weak-star in $L_\infty(Q_T)$. Finally, $\mathcal{L}_n[\boldsymbol{v}]\mathcal{L}_n[\nabla_x\varphi_3] \to \mathcal{L}[\boldsymbol{v}\cdot\nabla_x\varphi_3]$ weakly in $L_2(Q_T)$ due to the assertions 2, 7 in Proposition 2.6.

From (3.23) on the base of just established limiting relations we obtain that

$$\int_0^\tau dt \int_\Omega d\boldsymbol{x}[(\partial_t\varphi_1(t)\mathcal{L}[\varphi_3](\boldsymbol{x},t) + \varphi_1(t)\mathcal{L}[\boldsymbol{v}\cdot\nabla_x\varphi_3](\boldsymbol{x},t))\int_{S^1}\varphi_2(y)d\lambda_{t,x}(y)$$
$$+\sum_{i,j=1}^2 \varphi_1(t)\mathcal{L}[\varphi_3](\boldsymbol{x},t)U_{ij}(\boldsymbol{x},t)\int_{S^1} Y_{ij}\partial_y\varphi_2(y)d\lambda_{t,x}(y)]$$
$$= \int_\Omega d\boldsymbol{x}\,\varphi_1(\tau)\mathcal{L}[\varphi_3](\boldsymbol{x},t)\int_{S^1}\varphi_2(y)d\lambda_{\tau,x}(y) - \int_\Omega d\boldsymbol{x}\,\varphi_1(0)\varphi_3(\boldsymbol{x})\int_{S^1}\varphi_2(y)d\lambda_{0,x}(y). \quad (3.24)$$

In view of Lemma 3.3 it occurs that $\mathcal{L}^{-1}[\langle\lambda,\varphi_2\rangle](\boldsymbol{x},t) = \langle\nu_{t,x},\varphi_2\rangle$ a. e. in Q_T. Hence, $\langle\lambda_{t,x},\varphi_2\rangle = \mathcal{L}[\langle\nu,\varphi_2\rangle](\boldsymbol{x},t)$ a. e. in Q_T, where measure $\nu \in L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))$ is defined in the formulation of Lemma 3.3.

In the strength of Propositions 1.8 and 2.6 the equality (3.24) takes form

$$\int_0^\tau dt \int_\Omega \mathcal{L}[(\partial_t\varphi_1\varphi_3 + \varphi_1\boldsymbol{v}\cdot\nabla_x\varphi_3)\int_{S^1}\varphi_2(y)d\nu(y)$$
$$+\sum_{i,j=1}^2 \varphi_1\varphi_3\partial_i v_j\int_{S^1} Y_{ij}\partial_y\varphi_2(y)d\nu(y)](\boldsymbol{x},t)d\boldsymbol{x}$$
$$= \int_\Omega \varphi_1(\tau)\mathcal{L}\left[\varphi_3\int_{S^1}\varphi_2(y)d\nu(y)\right](\boldsymbol{x},\tau)d\boldsymbol{x} - \int_\Omega d\boldsymbol{x}\varphi_1(0)\varphi_3(\boldsymbol{x})\int_{S^1}\varphi_2(y)d\nu_{0,x}(y). \quad (3.25)$$

Basing on the assertion 2 in Proposition 1.8 we finally deduce the equality

$$\int_0^\tau dt \int_{\Omega\times S^1}[\partial_t\varphi_1(t)\varphi_2(y)\varphi_3(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla_x\varphi_3(\boldsymbol{x})\varphi_1(t)\varphi_2(y)$$
$$+\varphi_1(t)\partial_y\varphi_2(y)\varphi_3(\boldsymbol{x})(\nabla_x\boldsymbol{v} : Y)]\,d\mu_t(\boldsymbol{x},y)$$
$$= \int_{\Omega\times S^1}\varphi_1(\tau)\varphi_2(y)\varphi_3(\boldsymbol{x})d\mu_\tau(\boldsymbol{x},y) - \int_{\Omega\times S^1}\varphi_1(0)\varphi_2(y)\varphi_3(\boldsymbol{x})d\mu_0(\boldsymbol{x},y), \quad (3.26)$$

Here, in line with Definition 3.1, we define $d\mu_t(\boldsymbol{x},y) = d\boldsymbol{x}d\nu_{t,x}(y)$.

Observe that the linear span of the set of functions $\{\varphi_1(t)\varphi_2(y)\varphi_3(\boldsymbol{x}) \mid \varphi_1 \in C^1[0,T],\ \varphi_2 \in C^1(S^1), \varphi_3 \in C_0^1(\Omega)\}$ is dense in the set of finite in Ω functions from $C^1([0,T]\times\Omega\times S^1)$. Therefore (3.26) implies the equality (3.1). This means that μ_t is a measure-valued solution of Cauchy problem for Equation (1.5). ▲

4 Proof of Theorem 1.4

In the strength of Theorem 1.9 for verification of Theorem 1.1 it is enough to show that there exists a unique measure-valued solution (in the sense of Definition 3.2) of Cauchy problem for Equation (1.7) provided with initial data defined in the hypothesis in Theorem 1.4.

4.1 Existence of solution

Let $U_\varepsilon \in C^1(Q_T)$ and $\eta_0^\varepsilon \in C^1(\Omega \times S^1)$ are smooth regularizations of data provided for Equation (1.7) such that $\eta_0^\varepsilon \geq 0$ in $\Omega \times S^1$ and

$$U_\varepsilon \to U \text{ in } L_2(Q_T), \quad \eta_0^\varepsilon \to \eta_0 \text{ weak-star in } L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1)). \tag{4.1}$$

Remark, that such a choice of regularizations is both consistent and clear in view of the theory of distributions [29, Ch.II, §7.9]. Consider the regularized problem

$$\begin{array}{ll} \partial_t \eta_t^\varepsilon + \partial_y(\eta_t^\varepsilon\, Y \;:\; U_\varepsilon) = 0, & (t, \boldsymbol{x}, y) \in [0,T] \times \Omega \times S^1, \\ \eta_{t\varepsilon}(\boldsymbol{x}, y)|_{t=0} = \eta_0^\varepsilon(\boldsymbol{x}, y), & (\boldsymbol{x}, y) \in \Omega \times S^1. \end{array} \tag{4.2}$$

Due to the theory of linear PDEs of the first order [23, §§4–5] this problem has a solution $\eta_t^\varepsilon(\boldsymbol{x}, y) \in C^1([0,T] \times \Omega \times S^1)$ which obviously admits the decomposition $d\eta_t^\varepsilon(\boldsymbol{x}, y) = d\boldsymbol{x} d\lambda_{t,x}^\varepsilon(y)$, where $\lambda^\varepsilon \in L_2(0,T; L_{2,\mathrm{w}}(\Omega, M(S^1)))$ because η_t^ε is in sufficiently regular class. Multiplying both sides of (4.2) by a function $\Phi \in C^1([0,T] \times \Omega \times \mathbb{R})$ and integrating with respect to t and y over $S^1 \times [0,\tau]$, $\tau \leq T$, we obtain

$$\begin{aligned} \int_0^\tau dt \int_{S^1} (\partial_t \Phi + (U_\varepsilon \;:\; Y)\partial_y \Phi) d\lambda_{t,x}^\varepsilon(y) = \int_{S^1} \Phi(\tau, \boldsymbol{x}, y) d\lambda_{t,x}^\varepsilon(y) \\ - \int_{S^1} \Phi(0, \boldsymbol{x}, y) d\lambda_{0,x}^\varepsilon(y) \text{ far all } \boldsymbol{x} \in \Omega. \end{aligned} \tag{4.3}$$

Consider the Cauchy problem for the dual equation of (4.2).

$$\begin{array}{ll} \partial_t \Phi_\varepsilon + (Y \;:\; U_\varepsilon)\partial_y \Phi_\varepsilon = 0, & (t, \boldsymbol{x}, y) \in [0,T] \times \Omega \times S^1, \\ \Phi_\varepsilon|_{t=\tau} = \Phi(\tau, \boldsymbol{x}, y), & (\boldsymbol{x}, y) \in \Omega \times S^1. \end{array} \tag{4.4}$$

Recall [23, §§4–5] that it has a unique solution Φ which is in class $C^1([0,T] \times \Omega \times S^1)$ and has a form $\Phi_\varepsilon(t, \boldsymbol{x}, y) = \Phi(\tau, \mathbb{U}^{t,\tau}(\boldsymbol{x}, y))$, where $\mathbb{U}^{t,\tau}$ is a shift operator defined by the identity $\mathbb{U}^{t,\tau}(\boldsymbol{x}, y) = \boldsymbol{\varphi}(\boldsymbol{x}, y, s)|_{s=t}$. Here, $\boldsymbol{\varphi}$ is a solution of the Cauchy problem for the system of ordinary differential equations

$$\begin{array}{lll} (d/ds)\varphi_1 = 0, & (d/ds)\varphi_2 = 0, & (d/ds)\varphi_3 = U_\varepsilon(\varphi_1, \varphi_2, s) : Y(\varphi_3), \\ \varphi_1|_{s=\tau} = x_1, & \varphi_2|_{s=\tau} = x_2, & \varphi_3|_{s=\tau} = y, \quad y \in S^1. \end{array}$$

Let $\Phi(\tau, \boldsymbol{x}, y)$ in $[0,T] \times \Omega \times \mathbb{R}$ is greater than or equal to zero. The solution of the problem (4.4) provided with Φ standing for Cauchy data is nonnegative. That is, $\Phi_\varepsilon(t, \boldsymbol{x}, y) \geq 0$ in $[0,T] \times \Omega \times S^1$. Introducing this function into (4.3) we establish that

$$\int_{S^1} \Phi(\tau, \boldsymbol{x}, y) d\lambda_{t,x}^\varepsilon(y) = \int_{S^1} \Phi_\varepsilon(0, \boldsymbol{x}, y) d\lambda_{0,x}^\varepsilon(y) \geq 0$$

since $\lambda_{0,x}^\varepsilon$ is a nonnegative measure for all $\boldsymbol{x} \in \Omega$. It follows from this bound that $\lambda_{t,x}^\varepsilon$ is a nonnegative measure for all $(\boldsymbol{x}, t) \in Q_T$ and all $\varepsilon > 0$.

Let us obtain an estimate for the norm of the measure λ^ε. Introducing the test function $\Phi \equiv 1$ in (4.3) we get $\langle \lambda^\varepsilon_{\tau,x},\, 1\rangle = \langle \lambda^\varepsilon_{0,x},\, 1\rangle$. Hence $\|\lambda^\varepsilon_t\|_{L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1))} = \|\lambda^\varepsilon_0\|_{L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1))}$ $\forall\, t \in [0,T]$. Taking into account the limiting relation (4.1) we obtain

$$\|\lambda^\varepsilon\|_{L_2(0,T;L_{2,\mathrm{w}}(\Omega,\mathcal{M}_+(S^1)))} \le \mathrm{C}_0, \tag{4.5}$$

where C_0 is independent of ε and depends only on the norm of λ_0 in $L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^1))$.

Integration of (4.3) with respect to $\boldsymbol{x}$ over Ω leads to the equality

$$\begin{aligned}\int_0^\tau dt \int_\Omega d\boldsymbol{x} \int_{S^1} \Big(\partial_t \Phi + (U_\varepsilon(\boldsymbol{x},t) \,:\, Y)\partial_y \Phi\Big) d\lambda^\varepsilon_{t,x}(y) \\ = \int_\Omega d\boldsymbol{x} \int_{S^1} \Phi(\tau,\boldsymbol{x},y) d\lambda^\varepsilon_{\tau,x}(y) - \int_\Omega d\boldsymbol{x} \int_{S^1} \Phi(0,\boldsymbol{x},y) d\lambda^\varepsilon_{0,x}(y)\end{aligned} \tag{4.6}$$

valid for all $\Phi \in C^1([0,T] \times \Omega \times S^1)$.

Due to the bound (4.5) in the strength of Alaoglu theorem on weak-star precompactness of bounded sequences of linear functionals [30, I.3.12], the set $\{\lambda^\varepsilon\}_{\varepsilon>0}$ contains a subsequence such that

$$\lambda^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \lambda \quad \text{weak-star in } L_2(Q_T, \mathcal{M}_+(S^1)). \tag{4.7}$$

In view of (4.1) and (4.7) we can pass to the limit in (4.6) as $\varepsilon \to 0$. Thus, we conclude that (3.2) holds true, and hence the existence assertion in Theorem 1.4 is proved.

4.2 Uniqueness of solution

We base our proof of the uniqueness assertion on the following proposition.

Proposition 4.1. *Suppose that Cauchy problem for Equation* (1.7) *is provided with (not necessarily nonnegative) Cauchy data* $\eta_t|_{t=0} = \eta_0$, $d\eta_0(\boldsymbol{x},y) = d\boldsymbol{x} d\lambda_{0,x}(y)$, *and has a (not necessarily nonnegative) measure-valued solution* $\eta_t(\boldsymbol{x},y)$. *That is* $\eta_t(\boldsymbol{x},y)$ *satisfies Definition 3.2 in the formulation of which the non-negativeness condition is omitted.*

Then for all $\tau \in [0,T]$ *the equality*

$$\begin{aligned}\int_0^\tau dt \int_{S^1} \Big(\partial_t h + (U(\boldsymbol{x},t) \,:\, Y)\partial_y h\Big) d\lambda_{t,x}(y) \\ = \int_{S^1} h(\boldsymbol{x},y,\tau) d\lambda_{\tau,x}(y) - \int_{S^1} h(\boldsymbol{x},y,0) d\lambda_{0,x}(y)\end{aligned} \tag{4.8}$$

holds true for a. e. $\boldsymbol{x} \in \Omega$ *for any test function* $h(\boldsymbol{x},y,t)$ *such that* $h(\boldsymbol{x},y,t)$ *is measurable in* $\Omega \times S^1 \times (0,T)$, $\partial_y h(\boldsymbol{x},\cdot,\cdot) \in C(S^1 \times [0,T])$, *and* $\partial_t h(\boldsymbol{x},\cdot,\cdot) \in L_2(0,T; C(S^1))$ *for a. e.* $\boldsymbol{x} \in \Omega$.

PROOF. Introducing into (3.2) a test function of a form $\Phi(\boldsymbol{x}, y, t) = g(\boldsymbol{x})\bar{h}(\boldsymbol{x}, y, t)$, where $g \in L_\infty(\Omega)$, $\bar{h} \in L_\infty(\Omega, C^1(S^1 \times [0,T]))$, we arrive at the equality

$$\int_\Omega g d\boldsymbol{x} \int_0^\tau dt \int_{S^1} \Big(\partial_t \bar{h} + (U(\boldsymbol{x},t) : Y)\partial_y \bar{h}\Big) d\lambda_{t,x}(y)$$
$$= \int_\Omega g\, d\boldsymbol{x} \int_{S^1} \bar{h}(\boldsymbol{x}, y, \tau) d\lambda_{\tau,x}(y) - \int_\Omega g\, d\boldsymbol{x} \int_{S^1} \bar{h}(\boldsymbol{x}, y, 0) d\lambda_{0,x}(y). \quad (4.9)$$

Since g is an arbitrary function, it follows from (4.9) that for all $\tau \in [0,T]$ and for a. e. $\boldsymbol{x} \in \Omega$ the equality (4.8) holds true for any test function $\bar{h} \in L_\infty(\Omega, C^1(S^1 \times [0,T]))$. Hence, in order to conclude the justification of the proposition it is sufficient to show that for any function $h(\boldsymbol{x}, y, t)$ satisfying the hypothesis in the proposition there exists a sequence $\{h^k\} \subset L_\infty(\Omega, C^1(S^1 \times [0,T]))$ which has the following properties:

$$h^k \to h \qquad \text{a. e. in } \Omega \times S^1 \times (0,T), \quad (4.10)$$
$$h^k(\boldsymbol{x}, \cdot, \cdot) \to h(\boldsymbol{x}, \cdot, \cdot) \qquad \text{in } C(0,T; C^1(S^1)) \text{ and}$$
$$\partial_t h^k(\boldsymbol{x}, \cdot, \cdot) \to \partial_t h(\boldsymbol{x}, \cdot, \cdot) \qquad \text{in } L_2(0,T; C(S^1)) \text{ for a. e. } \boldsymbol{x} \in \Omega. \quad (4.11)$$

Let us consider the total orthogonal in $L_2(0, 2\pi)$ and $L_2(0,T)$ systems of trigonometric functions $\{\varphi_i(y)\}_{i=0}^\infty$ and $\{\psi_i(t)\}_{i=0}^\infty$, respectively. Assume

$$S_k(\boldsymbol{x}, y, t) = \sum_{i,j=0}^{k} c_{ij}(\boldsymbol{x})\varphi_i(y)\psi_j(t),\ k = 1, 2, \ldots$$

is a sequence of the partial Fourier sums of a function $h(\boldsymbol{x}, y, t)$, where

$$c_{ij}(\boldsymbol{x}) = \frac{2}{T\pi} \int\limits_{(0,2\pi)\times(0,T)} h(\boldsymbol{x}, y, t)\varphi_i(y)\psi_j(t) dy dt,\ i, j = 1, 2, \ldots, k$$

are Fourier coefficients. Assume

$$C_k(\boldsymbol{x}, y, t) = \frac{S_0(\boldsymbol{x}, y, t) + \ldots + S_k(\boldsymbol{x}, y, t)}{k+1},\ k = 1, 2, \ldots, k$$

is a sequence of the arithmetic means associated with the Fourier series of $h(\boldsymbol{x}, y, t)$.

It is easy to see that $C_k(\boldsymbol{x}, \cdot, \cdot) \in C^1(S^1 \times [0,T])$ for a. e. $\boldsymbol{x} \in \Omega$, $k \geq 1$, and the expression $h(\boldsymbol{x}, \cdot, \cdot)\varphi_i(\cdot)\psi_j(\cdot)$ is integrable on $S^1 \times (0,T)$ for a. e. $\boldsymbol{x} \in \Omega$ [30, Ch.I, §4]. Hence, coefficients $c_{ij}(\boldsymbol{x})$ are measurable in Ω, consequently, C_k are measurable in $\Omega \times S^1 \times (0,T)$. Due to Luzin theorem for any n there exists a closed set $\overline{\Omega}_n^{ij}$, $\mathtt{meas}\,\overline{\Omega}_n^{ij} > \mathtt{meas}\,\Omega - n^{-1}$ such that function $c_{ij}(\boldsymbol{x})$ is continuous in $\overline{\Omega}_n^{ij}$ $(i, j = 1, 2, \ldots)$. Define

$$S_l^k = \sum_{i,j=0}^{l} c_{ij}^{k^4}(\boldsymbol{x})\varphi_i(y)\psi_j(t), \quad h^k = \frac{S_0^k + \ldots + S_k^k}{k+1},\ \text{where}$$

$$c_{ij}^n(\boldsymbol{x}) = \begin{cases} c_{ij}(\boldsymbol{x}), & \boldsymbol{x} \in \overline{\Omega}_n^{ij}, \\ 0, & \boldsymbol{x} \in \Omega \setminus \overline{\Omega}_n^{ij}, \end{cases} \quad i,j = 0,1,\dots;\ n = 1,2,\dots.$$

Notice that $h^k \in L_\infty(\Omega; C^1(S^1 \times [0,T]))$, $k = 1,2,\dots$;

$$h^k(\boldsymbol{x}, y, t) = C_k(\boldsymbol{x}, y, t) \text{ for } \boldsymbol{x} \in \cap_{i,j=1}^k \overline{\Omega}_{k^4}^{ij},$$

$$\mathtt{meas}\ \cap_{i,j=1}^k \overline{\Omega}_{k^4}^{ij} \geq \mathtt{meas}\,\Omega - \sum_{i,j=1}^k \mathtt{meas}(\Omega \setminus \overline{\Omega}_{k^4}^{ij}) = \mathtt{meas}\,\Omega\ - k^{-2},$$

and for any $n \in \mathbb{N}$ the bound

$$\mathtt{meas} \bigcap_{k=n}^{\infty} \left(\bigcap_{i,j=1}^{k} \overline{\Omega}_{k^4}^{ij} \right) \geq \mathtt{meas}\,\Omega\ - \sum_{k=n}^{\infty} k^{-2} = \mathtt{meas}\,\Omega - n^{-1}$$

is valid. Denoting $\overline{\Omega}^n = \bigcap_{k=n}^\infty \left(\bigcap_{i,j=1}^k \overline{\Omega}_{k^4}^{ij} \right)$ we obtain that

$$h^k(\boldsymbol{x}, y, t) = C_k(\boldsymbol{x}, y, t) \text{ for } \boldsymbol{x} \in \overline{\Omega}_n \ \forall\, k \geq n.$$

On the other hand, $C_k(\boldsymbol{x}, \cdot, \cdot) \to h(\boldsymbol{x}, \cdot, \cdot)$ uniformly on $S^1 \times [0,T]$ and strongly in $C(0,T; C^1(S^1))$, and $\partial_t C_k(\boldsymbol{x}, \cdot, \cdot) \to \partial_t h(\boldsymbol{x}, \cdot, \cdot)$ in $L_2(0,T; C(S^1))$ for a. e. $\boldsymbol{x} \in \Omega$ due to Fejér theorem and its consequence for summable functions [8, Ch.5, §3.1; Ch.6, §1.1]. Thus,

$$\begin{array}{ll} h^k \to h & \text{a. e. in } \overline{\Omega}_n \times S^1 \times [0,T], \\ h^k(\boldsymbol{x}, \cdot, \cdot) \to h(\boldsymbol{x}, \cdot, \cdot) & \text{strongly in } C(0,T; C^1(S^1)) \\ & \text{and uniformly in } S^1 \times [0,T]\ \forall\, \boldsymbol{x} \in \overline{\Omega}_n, \\ \partial_t h^k(\boldsymbol{x}, \cdot, \cdot) \to \partial_t h(\boldsymbol{x}, \cdot, \cdot) & \text{in } L_2(0,T; C(S^1)). \end{array}$$

Since $n \in \mathbb{N}$ is arbitrary and the measure of the set $\overline{\Omega} \setminus \overline{\Omega}_n$ is bounded from above by $1/n$ these limiting relations show that (4.10), (4.11) are valid for the constructed sequence $\{h^k\}_{k=1}^\infty$. ▲

Let us turn back to verification of the uniqueness assertion in Theorem 1.4.

Suppose that η_t' and η_t'' are two measure-valued solutions (in the sense of Definition 3.2) of Cauchy problem for Equation (1.7) provided with initial Cauchy data such that $\eta_t'|_{t=0} = \eta_t''|_{t=0}$. Since Equation (1.7) is linear, the (not necessarily nonnegative) measure $\eta_t = \eta_t' - \eta_t''$ solves Cauchy problem for (1.7) provided with zero initial data. Thus, in order to complete justification of uniqueness assertion one needs to establish that η_t is zero measure. It amounts to showing that if Cauchy data η_0 satisfy $d\eta_0 = d\boldsymbol{x} d\lambda_{0,x}$ and the identity

$$\int_{S^1} f(y) d\lambda_{0,x}(y) = 0 \text{ for a. e. } \boldsymbol{x} \in \Omega\ \forall\, f \in C(S^1), \tag{4.12}$$

then a solution η_t of Cauchy problem for Equation (1.7) satisfying $d\eta_t(\boldsymbol{x}, y) = d\boldsymbol{x}\, d\lambda_{t,x}(y)$ admits the equality

$$\int_{S^1} f(y) d\lambda_{t,x}(y) = 0 \text{ for a. e. } (\boldsymbol{x}, t) \in Q_T. \tag{4.13}$$

In the strength of Proposition 4.1 $\lambda_{t,x}$ satisfies the equality (4.8) which has a form

$$\int_0^\tau dt \int_{S^1} \Big(\partial_t h + (U(\boldsymbol{x},t) : Y)\partial_y h\Big) d\lambda_{t,x}(y) = \int_{S^1} h(\boldsymbol{x},y,\tau) d\lambda_{\tau,x}(y) \tag{4.14}$$

due to the formula (4.12).

Consider the following Cauchy problem which depends on $\boldsymbol{x}$ as on a parameter.

$$\begin{aligned} &\partial_t h(\boldsymbol{x},y,t) + (U(\boldsymbol{x},t) : Y)\partial_y h(\boldsymbol{x},y,t) = 0, \quad (\boldsymbol{x},y,t) \in \Omega \times S^1 \times [0,T], \\ &h(\boldsymbol{x},y,t)|_{t=\tau} = h_\tau(\boldsymbol{x},y) = h_1(\boldsymbol{x})h_2(y)h_3(\tau), \quad (\boldsymbol{x},y,\tau) \in \Omega \times S^1 \times [0,T]. \end{aligned} \tag{4.15}$$

Here, we assume $h_1 \in C^1(\Omega)$, $h_2 \in C^1(S^1)$ and $h_3 \in C^1[0,T]$. The equation (4.15) is understood in the sense of the theory of distributions.

Our aim is to show that a solution of this problem is a legitimal test function for the integral equality (4.8). Thus, we will conclude the verification of the uniqueness assertion of the theorem since introducing of such a function into (4.8) leads to the identity

$$h_1(\boldsymbol{x})h_3(\tau) \int_{S^1} h_2(y) d\lambda_{\tau,x}(y) = 0 \text{ for a. e. } \boldsymbol{x} \in \Omega,$$

and, consequently, to the equality (4.13) due to arbitrariness of the values of $\tau \in [0,T]$ and forms of the functions h_1, h_2, h_3.

In order to prove the solvability of (4.15) fix $\boldsymbol{x}^* \in \Omega$ such that $U(\boldsymbol{x}^*,\cdot) \in L_2(0,T)$. In the strength of Fubini theorem [30, I.4.45] the set of such x^* has the total Lebesgue measure in Ω. In view of the theory of linear PDEs [23, §§4–5], a solution of (4.15) is given by formula $\tilde{h}(\boldsymbol{x}^*,y,t) = h_\tau(\boldsymbol{x}^*,\varphi(\boldsymbol{x}^*,y,t))$, where $\varphi(\boldsymbol{x}^*,y,t)$ is a solution of the Cauchy problem

$$(d/ds)\varphi = U(\boldsymbol{x}^*,s) : Y(\varphi), \ s \in [0,T], \quad \varphi|_{s=\tau} = y, \ y \in S^1. \tag{4.16}$$

Due to Carathéodory theorem [11, Ch.2, §5.3] a solution of (4.16) exists, is unique, is absolutely continuous with respect to s on interval $(0,T)$, and continuously differentiable with respect to Cauchy data $y \in S^1$ since $U(\boldsymbol{x}^*,t) \in L_2(0,T)$, $Y \in C^\infty(S^1)$. The proof of continuously differentiability does not differ from the one given in [21, Ch.I, §5] in the case of continuous with respect to s right hand side of (4.16). It is quite clear that $\tilde{h}$ has the same regularity properties as φ.

Multiplying (4.15) by an arbitrary function $w \in C^1[0,T]$, such that $w(0) = w(T) = 0$ and integrating with respect to t we arrive at the equality

$$\int_0^T (\partial_t w)\tilde{h} dt = \int_0^T w(U(\boldsymbol{x},t) : Y)\partial_y \tilde{h}\, dt. \tag{4.17}$$

In line with the definition of generalized derivation with respect to t it follows from (4.17) that $\partial_t \tilde{h}(\boldsymbol{x}^*) \in L_2(0,T;C(S^1))$, because $(U(\boldsymbol{x}^*) : Y)\partial_y \tilde{h}(\boldsymbol{x}^*) \in L_2(0,T;C(S^1))$.

Thus, we conclude that $\tilde{h}(\boldsymbol{x})$ is continuous in $S^1 \times [0,T]$, $\partial_t \tilde{h}(\boldsymbol{x}) \in L_2(0,T;C(S^1))$, and $\partial_y \tilde{h}(\boldsymbol{x}) \in C([0,T] \times S^1)$ for a. e. $\boldsymbol{x} \in \Omega$.

At the end, it remains to establish measurability of $\tilde{h}$ with respect to $(\boldsymbol{x}, y, t)$ in $\Omega \times S^1 \times (0,T)$ which amounts to verify measurability of the solution of (4.16). Due to Luzin theorem for any $\varepsilon > 0$ there exists a closed set $\overline{Q}_T^{\varepsilon} \subset Q_T$, such that $\mathtt{meas}\, \overline{Q}_T^{\varepsilon} \geq \mathtt{meas}\, Q_T - \varepsilon$, and $U(\boldsymbol{x}, s)$ is continuous in $\overline{Q}_T^{\varepsilon}$. In view of Carathéodory theorem the solution $\varphi(\boldsymbol{x}, y, s)$ of the problem (4.16) is in $C(\overline{Q}_T^{\varepsilon} \times S^1)$. Hence, it satisfies the hypothesis in Luzin theorem, and, consequently, is measurable in $\Omega \times S^1 \times (0,T)$. ▲

4.3 Appendix. Generalization to any space dimension N

The restriction to the case of dimension two is not fundamental, i.e. arguments in the paper can be generalized (in a natural way) to any space dimension N. Consequently, the following theorem, similar to theorem 1.4, holds true.

Theorem 4.1. *If $\boldsymbol{v} \in L_2(0,T; \dot{J}^1(\Omega))$ and the non-negative measure μ_0 defined on $\Omega \times S^{N-1}$ is such that*

$$d\mu_0(\boldsymbol{x}, y_1, \ldots, y_{N-1}) = d\boldsymbol{x} d\nu_{0,x}(y_1, \ldots, y_{N-1}), \quad \nu_0 \in L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^{N-1})),$$

then the Cauchy problem for the Tartar equation

$$\partial_t \mu_t + \boldsymbol{v} \cdot \nabla_x \mu_t + \sum_{k=1}^{N-1} \partial_{y_k} (\mu_t \, Y_k \ : \nabla_x \boldsymbol{v}) = 0,$$
$$t \in [0,T], \ \boldsymbol{x} \in \Omega, \ (y_1, \ldots, y_{N-1}) \in S^{N-1},$$

with Cauchy data $\mu_t|_{t=0} = \mu_0$, has a unique non-negative measure-valued solution μ_t such that

$$d\mu_t(\boldsymbol{x}, y_1, \ldots, y_{N-1}) = d\boldsymbol{x} d\nu_{t,x}(y_1, \ldots, y_{N-1}),$$
$$\nu \in L_2(0,T; L_{2,\mathrm{w}}(\Omega, \mathcal{M}_+(S^{N-1}))).$$

Here, Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary, S^{N-1} is the unit sphere in $\mathbb{R}^N$, $y_1, \ldots, y_{N-1}$ are angular coordinates on S^{N-1}, and

$$Y_k = Y_k(y_1, \ldots, y_{N-1}), \quad k = 1, \ldots, N-1,$$

are (N-1)x(N-1) matrices consisting of known infinitely smooth components $Y_k^{ij}(y_1, \ldots, y_{N-1})$. The explicit forms of these components depend on a choice of parametrization on S^{N-1}.

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