Solutions of a Problem on Motion of Viscous Incompressible Fluid Provided with Frequently Oscillating Initial Data*

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Abstract

In the present article, an initial-boundary value problem for the system of the classical Navier–Stokes equations describing dynamics of non-homogeneous viscous incompressible fluid in a bounded domain $\Omega \subset R^2$ is considered. It is supposed that the values of density $\rho(x,t)$ are translated along the trajectories of motions of fluid particles with the velocity $\vec{v}(x,t)$. The well posedness of this problem was established in [1]. We investigate the weak limits of sequences of solutions $\rho_\varepsilon(x,t)$, $\vec{v}_\varepsilon(x,t)$, $\nabla p_\varepsilon(x,t)$ of this problem as $\varepsilon \to 0$, provided with initial distributions of density $\rho_\varepsilon(x,0)$ and velocity $\vec{v}_\varepsilon(x,0)$ satisfying the following conditions:

$$\rho_\varepsilon(x,0) \to \rho_0(x) \quad \text{weakly* in } L_\infty(\Omega), \quad \vec{v}_\varepsilon(x,0) \to \vec{v}_0(x) \text{ in } H^1(\Omega).$$

We prove that, in this case, $\rho_\varepsilon \to \rho$ weakly* in $L_\infty(\Omega \times [0,T])$, $\vec{v}_\varepsilon \to \vec{v}$ weakly in $L_2(0,T;H^2(\Omega))$, and $\nabla p_\varepsilon \to \nabla p$ weakly in $L_2(\Omega \times [0,T])$ as $\varepsilon \to 0$, where the triple of functions $\vec{v}(x,t)$, $\rho(x,t)$ and $\nabla p(x,t)$ is a strong generalized solution of the problem under consideration provided with the initial data $\vec{v}_0(x)$ and $\rho_0(x)$. Besides, we establish that the Tartar H-measure μ [2], associated with the extracted subsequence $\{\rho_\varepsilon(x,t)\}$, is a solution of the transport equation

$$D_t\mu + \operatorname{div}_x(\mu\,\vec{v}) + \frac{\partial}{\partial y}(\mu\, Y \,:\nabla_x\vec{v}) = 0, \quad (t,x,y) \in [0,T] \times \Omega \times S^1,$$

in which

$$Y = \begin{pmatrix} -\frac{1}{2}\sin 2y & \cos^2 y \\ -\sin^2 y & \frac{1}{2}\sin 2y \end{pmatrix}.$$

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1 Existence of a solution to the problem

1.1 The statement of the problem and review of the known results

The following problem is under consideration.
Problem A. In a space-time cylinder $Q_T = \{(x,t) \in \Omega \times [0,T]\}$, where Ω is a bounded domain in R^2 and $T = \text{const } > 0$, find a velocity field $\vec{v}(x,t) = \{v_1(x,t),\ v_2(x,t)\}$, a density distribution $\rho = \rho(x,t)$ and a hydraulic pressure $p = p(x,t)$ satisfying the equations

$$\rho D_t \vec{v} + \rho \sum_{i=1}^{2} v_i D_i \vec{v} - \nu \Delta \vec{v} + \nabla p = \rho \vec{f}, \tag{1}$$

$$\text{div}\vec{v} = 0, \tag{2}$$

$$D_t \rho + \sum_{i=1}^{2} v_i D_i \rho = 0, \tag{3}$$

and initial and boundary conditions

$$\vec{v}|_{\partial\Omega} = 0, \quad \vec{v}(x,0) = \vec{v}_0(x), \tag{4}$$

$$\rho(x,0) = \rho_0(x). \tag{5}$$

In (1)–(5) and further in the paper, $D_t = \partial/\partial t$, $D_i = \partial/\partial x_i$, and viscosity ν is a given positive constant.
Definition 1. The triple of functions $\vec{v}(x,t)$, $\rho(x,t)$ and $\nabla p(x,t)$ is called a strong generalized solution if it satisfies the conditions $\vec{v} \in L_2(0,T;H^2(\Omega)) \cap L_\infty(0,T;H^1(\Omega))$, $D_t\vec{v} \in L_2(Q_T)$, $\nabla p \in L_2(Q_T)$, and $\rho \in L_\infty(Q_T)$, the equalities

$$\rho D_t \vec{v} + \rho \sum_{i=1}^{2} v_i D_i \vec{v} - \nu \Delta \vec{v} + \nabla p = \rho \vec{f}, \quad \text{almost everywhere in } Q_T, \tag{6}$$

$$\text{div}\vec{v} = 0, \quad \text{almost everywhere in } Q_T, \tag{7}$$

and the integral equality

$$\int_{Q_T} \rho(D_t\varphi + \sum_{i=1}^{2} v_i D_i \varphi) dxdt + \int_\Omega \rho_0 \varphi(x,0) dx = 0 \tag{8}$$

for any test function φ satisfying the conditions $\varphi(t) \in C^1([0,T],\ W_2^1(\Omega))$, and $\varphi(x,T) = 0$.

Recall that $H^k(\Omega)$, $k \in Z$ is the closure of the set of functions $\{\phi \mid \quad \phi \in C_0^\infty(\Omega), \quad \text{div}\phi = 0\}$ with respect to norm in $W_2^k(\Omega)$, and we denote $H^0 \equiv H$.

If $\vec{f} \in L_2(0,T,L_2(\Omega))$, $\vec{v}_0 \in H^1(\Omega)$ and $m \le \rho_0(x) \le M$, $x \in \Omega$, $0 < m, M < \infty$, $m,\ M = \text{const}$, then it is proved in [1] that the Galerkin approximations $\{\vec{v}^N,\ \rho^N\}$ admit the bounds

$$\max_{0\le t\le T} \|\vec{v}^N(t)\|_{H^1(\Omega)} \le \exp\left(C_1 M^2 [\frac{1}{2}\|\vec{v}_0\|^2_{2,\Omega} + \|\vec{f}\|_{2,1,Q_T}(\|\vec{f}\|_{2,1,Q_T} + \|\vec{v}_0\|_{2,\Omega})]^2 \right) [\|\vec{f}\|^2_{2,Q_T} + \|\vec{v}_0\|^2_{H^1(\Omega)}]; \tag{9}$$

$$\|D_t\vec{v}^N\|^2_{2,Q_T} \le \frac{\nu}{m}\left(C_2[\|\vec{f}\|_{2,Q_T} + \|\vec{v}_0\|^2_{H^1(\Omega)}]^2 + C_3\|\vec{f}\|_{2,Q_T} + \|\vec{v}_0\|^2_{H^1(\Omega)}\right); \tag{10}$$

$$\|\vec{v}^N_{xx}\|_{2,Q_T} \le C_4[\|\vec{f}\|^2_{2,Q_T} + \|\vec{v}_0\|^2_{H^1(\Omega)} + 1]; \tag{11}$$

and

$$m \le \rho^N(x,t) \le M. \tag{12}$$

In (9)–(12), $C_1 - C_4$ are constants that do not depend on the number N of the Galerkin approximation and on the given data in the problem.

It was also proved that these estimates (along with some other properties of the sequence $\{\vec{v}^N, \rho^N\}$, which were established in [1]) yield the following limit relations:

$$\vec{v}^N \to \vec{v} \quad \text{weakly in } L_2(0,T,H^2(\Omega)) \quad \text{and weakly* in } L_\infty(0,T;H^1(\Omega)), \tag{13}$$

$$D_t\vec{v}^N \to D_t\vec{v} \quad \text{weakly in } L_2(Q_T), \tag{14}$$

$$\rho^N \to \rho \quad \text{weakly* in } L_\infty(Q_T), \quad \text{and strongly in } L_q(Q_T),\ 1 \le q < \infty, \tag{15}$$

where the triple $\vec{v} \in L_\infty(0,T,H^1(\Omega) \cap L_2(0,T,H^2(\Omega))$, $\rho \in L_\infty(Q_T)$ and $\nabla p(x,t)$ is a strong generalized solution of Problem A.

1.2 A solution of Problem A provided with frequently oscillating initial data

Frequently oscillating initial data are modeled as the sequences of the initial distributions $\rho_\varepsilon(x,0)$ and $\vec{v}_\varepsilon(x,0)$ in Problem A, where $\varepsilon \to 0$. Consider

$$m \le \rho_{0\varepsilon} \le M, \quad \rho_{0\varepsilon}(x) \to \rho_0(x) \quad \text{weakly* in } L_\infty(\Omega), \tag{16}$$

$$\vec{v}_{0\varepsilon}(x) \to \vec{v}_0(x) \quad \text{in } H^1(\Omega). \tag{17}$$

According to Section 1.1, for any $\varepsilon > 0$ there exists a generalized solution $\{\vec{v}_\varepsilon(x,t),\ \rho_\varepsilon(x,t),\ \nabla p_\varepsilon(x,t)\}$, corresponding to the initial data $\{\vec{v}_{0\varepsilon}(x),\ \rho_{0\varepsilon}(x)\}$.

We prove the following

Theorem 1. *Let $\{\vec{v}_{0\varepsilon}(x),\ \rho_\varepsilon(x)\}$ satisfy the conditions (16) and (17). Then, there exists a sequence $\{\vec{v}_\varepsilon(x,t),\ \rho_\varepsilon(x,t),\ \nabla p_\varepsilon(x,t)\}$ of strong generalized solutions of Problem A corresponding to the initial data $\{\vec{v}_{0\varepsilon}(x),\ \rho_\varepsilon(x)\}$, and there exist functions $\vec{v}(x,t)$, $\rho(x,t)$ and $\nabla p(x,t)$ such that*

$$\vec{v}_\varepsilon(x,t) \to \vec{v}(x,t) \quad \text{weakly in } L_2(0,T,H^2(\Omega)),\ \text{weakly* in } L_\infty(0,T;H^1(\Omega)), \tag{18}$$

$$D_t\vec{v}_\varepsilon(x,t) \to D_t\vec{v}(x,t) \quad \text{weakly in } L_2(Q_T), \tag{19}$$

$$\rho_\varepsilon(x,t) \to \rho(x,t) \quad \text{weakly* in } L_\infty(Q_T), \tag{20}$$

$$\nabla p_\varepsilon(x,t) \to \nabla p(x,t) \quad \text{weakly in } L_2(Q_T); \tag{21}$$

the triple of functions $\vec{v}(x,t)$, $\rho(x,t)$, $\nabla p(x,t)$ is a strong generalized solution of Problem A provided with initial data $\vec{v}(x,0) = \vec{v}_0(x)$, $\rho(x,0) = \rho_0(x)$, where $\vec{v}(x,0) = \vec{v}_0(x)$ and $\rho(x,0) = \rho_0(x)$ are the weak limits of the sequences $\vec{v}_{0\varepsilon}(x)$ and $\rho_{0\varepsilon}(x)$ in $H^1(\Omega)$ and $L_\infty(\Omega)$, respectively.*

We prove the theorem in two steps. The first step consists in the justification of the following auxiliary

Lemma 1. *As $\varepsilon \to 0$, the sequence of strong generalized solutions of Problem A converges to a weak generalized solution of Problem A.*

Definition 2. A pair of functions $\vec{v} \in L_2(0,T;H^1(\Omega)) \cap L_\infty(0,T;H(\Omega))$ and $\rho \in L_\infty(Q_T)$, $0 < m \le \rho(x,t) \le M < \infty$ (almost everywhere in Q_T), is called a weak generalized solution of Problem A if it satisfies the following two integral equalities

$$\int_{Q_T} \rho\vec{v}\cdot(D_t\vec{\Phi} + \sum_{i=1}^{2} v_i D_i\vec{\Phi})dxdt - \int_{Q_T} \nu\nabla\vec{v} : \nabla\vec{\Phi}dxdt + \int_{Q_T} \rho\vec{f}\cdot\vec{\Phi}dxdt + \int_\Omega \rho_0\vec{v}_0\cdot\vec{\Phi}(x,0)dx = 0, \quad (22)$$

$$\int_{Q_T} \rho(D_t\varphi + \sum_{i=1}^{2} v_i D_i\varphi)dxdt + \int_\Omega \rho_0\varphi(x,0)dx = 0, \quad (23)$$

where $\vec{\Phi}$ and φ are test functions satisfying the conditions $\vec{\Phi}(t) \in C^1([0,T], H^1(\Omega))$, $\varphi(t) \in C^1[0,T,W_2^1(\Omega)]$, $\mathrm{div}\vec{\Phi} = 0$, $\vec{\Phi}|_{\partial\Omega} = 0$, $\vec{\Phi}(x,T) = 0$, and $\varphi(x,T) = 0$.
Proof of Lemma 1. From the bounds (9)–(12), we deduce the uniform with respect to ε estimates for $\vec{v}_\varepsilon(x,t)$, $\rho_\varepsilon(x,t)$:

$$\left(\|\vec{v}_\varepsilon\|_{L_2(0,T;H^2(\Omega))},\ \|\vec{v}_\varepsilon\|_{L_\infty(0,T;H^1(\Omega))},\ \|D_t\vec{v}_\varepsilon\|_{2,Q_T}\right) \le C_5, \quad (24)$$

$$m \le \rho_\varepsilon \le M, \quad (x,t) \in Q_T, \quad (25)$$

where C_5 depends only on $C_1 - -C_4$.

The estimates (24), (25) and the equality

$$\nabla p_\varepsilon = -\rho_\varepsilon D_t\vec{v}_\varepsilon - \rho_\varepsilon \sum_{i=1}^{2} v_{i\varepsilon} D_i\vec{v}_\varepsilon + \nu\Delta\vec{v}_\varepsilon + \rho_\varepsilon\vec{f}, \quad \text{a.e. in } Q_T, \quad (26)$$

immediately yield the formulae (18)–(21).

Introduce the Banach space $W = \{\vec{u}\,|\,\vec{u} \in L_2(0,T;H^2(\Omega)), \quad D_t\vec{u} \in L_2(0,T;H(\Omega))\}$, equipped with the norm

$$\|\vec{u}\|_W = \|\vec{u}\|_{L_2(0,T;H^2(\Omega))} + \|D_t\vec{u}\|_{L_2(0,T;H(\Omega))}.$$

The embedding of $H^2(\Omega)$ into $H^1(\Omega)$ is compact due to the Rellich theorem, $\vec{v}_\varepsilon \in W$ thanks to estimate (24). Hence, in the strength of the Lions compactness lemma [3], the following limit relation holds true:

$$\vec{v}_\varepsilon \to \vec{v} \text{ in } L_2(0,T;H^1(\Omega)), \quad (27)$$

consequently, since $H^1(\Omega) \subset L_4(\Omega)$, one also has

$$\vec{v}_\varepsilon \to \vec{v} \text{ in } L_2(0,T;L_4(\Omega)). \quad (28)$$

The formulae (16)–(18), (27), (28) provide the necessary basis for the limiting transitions as $\varepsilon \to 0$ in all terms in the integral equalities (22) and (23). Therefore, these equalities are valid with $\vec{v}_\varepsilon(x,t)$ and $\rho_\varepsilon(x,t)$ replaced by $\vec{v}(x,t)$ and $\rho(x,t)$, respectively. Lemma 1 is proved. □

The second step of the proof of Theorem 1 consists in verification of the assertion that the immediately above obtained weak generalized solution of Problem A is in fact a strong generalized solution to Problem A.

In the strength of the estimates (24) and (25) and the equality (26), the solution under consideration meets the requirements of regularity of strong generalized solutions:

$$\begin{cases} \vec{v} \in L_2(0,T;H^2(\Omega)) \cap L_\infty(0,T;H^1(\Omega)) \cap W_2^1(0,T;H(\Omega)), \\ \nabla p \in L_2(Q_T), \\ \rho \in L_\infty(Q_T), \quad 0 < m \le \rho(x,t) \le M < \infty \text{ a.e. in } Q_T. \end{cases} \tag{29}$$

The functions $\vec{v}(x,t)$ and $\rho(x,t)$ satisfy the integral equality (23) (or, equivalently, the equality (8)). In order to finish verification of Theorem 1, it remains to establish the equality (6) almost everywhere in Q_T.

Let a sequence $\{\vec{v}_n\} \subset C^1(Q_T)$ be such that

$$\vec{v}_n \to \vec{v} \text{ in } L_2(0,T;H^2(\Omega)) \cap L_\infty(0,T;H^1(\Omega)) \cap W_2^1(0,T;H(\Omega)), \tag{30}$$

$$\vec{v}_n(x,0) \to \vec{v}^0(x) \text{ in } H^1(\Omega), \quad \operatorname{div} \vec{v}_n = 0, \quad \vec{v}_n|_{\partial\Omega} = 0. \tag{31}$$

Represent the equality (22) in the form

$$\begin{aligned} \Big(\int_{Q_T} \rho(D_t(\vec{\Phi}\cdot\vec{v}_n) + \sum_{i=1}^{2} v_i D_i(\vec{\Phi}\cdot\vec{v}_n))dxdt + \int_\Omega \rho_0(\vec{v}_n(x,0)\cdot\vec{\Phi}(x,0))dx\Big) \\ - \left[\int_{Q_T} (\rho(D_t\vec{v}_n + \sum_{i=1}^{2} v_i D_i\vec{v}_n) - \nu\Delta\vec{v}_n - \rho\vec{f})\vec{\Phi})dxdt\right] \\ + \Big\{\int_{Q_T} \rho(\vec{v}-\vec{v}_n)(D_t\vec{\Phi} + \sum_{i=1}^{2} v_i D_i\vec{\Phi})dxdt - \int_{Q_T} \nu\nabla(\vec{v}-\vec{v}_n) : \nabla\vec{\Phi}dxdt \\ + \int_\Omega \rho_0(\vec{v}_0 - \vec{v}_n(x,0))\vec{\Phi}(x,0)dx\Big\} = 0. \end{aligned} \tag{32}$$

The product $\vec{\Phi}\cdot\vec{v}_n$ is a valid test function for the equality (23) since $\vec{\Phi}\cdot\vec{v}_n \in C^1(0,T;W_2^1(\Omega))$. This fact yields that the expression in the big brackets (i.e. in $(\ldots)$) in (32) is the identical zero. The formulae (29)–(31) and the conditions of Theorem 1 allow to proceed in (32) the limiting transition as $n \to \infty$. Within this limiting transition, the expression in the parenthesis in (32) tends to zero, and we obtain eventually the following:

$$\int_{Q_T} \left[\rho(D_t\vec{v} + \sum_{i=1}^{2} v_i D_i\vec{v}) - \nu\Delta\vec{v} - \rho\vec{f}\right] \vec{\Phi}dxdt = 0. \tag{33}$$

Here, $\vec{\Phi}$ is a solenoidal function in $C^1(0,T;H^1(\Omega))$, and the expression in the square bracketsin (33) belongs to $L_2(Q_T)$. Moreover, this expression is the gradient of some function almost everywhere in Q_T in the strength of the arbitrariness of Φ and the well known theorem on the decomposition of the space L_2. Theorem 1 is proved. □

2 The Tartar H-measure. The theorem on continuation of the measure

In [2], there was proposed the notion of H-measure, which effectively describes the evolution of oscillatory phenomena in problems with highly oscillatory initial data. The rest of the present paper is devoted to a studying of H-measures associated with solutions of Problem A.

Let A: $L_2(R^2) \to L_2(R^2)$ be a pseudo-differential operator of zero order with the principal symbol $a(\frac{\xi}{|\xi|})$, $a \in C^1(S^1)$, $\xi \in R^2$, which means in terms of the Fourier transform F that $F[A[u]](\xi) = a(\frac{\xi}{|\xi|})F[u](\xi)$, where $u \in L_2(R^2)$, and the Fourier transform is defined by $F[u](\xi) = \int_{R^2} e^{2\pi i x\xi} u(x)dx$. Let $\varphi_1, \varphi_2 \in C_0(\Omega)$, $\psi_\varepsilon \to \psi$ weakly in $L_2(\Omega)$, and φ_1, φ_2, $\psi_\varepsilon(x)$, and $\psi(x)$ are equal to zero outside Ω.

Choosing (if necessary) a subsequence from ψ_ε, define the mapping μ: $C(S^1) \times C_0(\Omega) \times C_0(\Omega) \to R$ by means of the formula

$$\langle \mu,\ a\varphi_1\varphi_2 \rangle = \lim_{\varepsilon\to 0} \int_{R^2} \varphi_1(\psi_\varepsilon - \psi) A[\varphi_2(\psi_\varepsilon - \psi)]dx, \tag{34}$$

or, equivalently, in terms of the Fourier transform

$$\langle \mu,\ a\varphi_1\varphi_2 \rangle = \lim_{\varepsilon\to 0} \int_{R^2} F[\varphi_1(\psi_\varepsilon - \psi)](\xi)\ a(\frac{\xi}{|\xi|})\overline{F[\varphi_2(\psi_\varepsilon - \psi)](\xi)}d\xi.$$

In [2], the following fundamental theorem was proved:

Theorem 2. *The mapping μ is a non-negative Borelian measure in $\Omega \times S^1$.*

Remark. In view of this theorem, the formally above introduced in (34) duality brackets make in fact the ordinary sense of an integral with respect to measure μ:

$$\langle \mu,\ f \rangle = \int_{\Omega\times S^1} f\, d\mu, \quad f \in C(\Omega \times S^1).$$

Definition 3. *Measure μ is called the H-measure associated with the (sub)sequence $\psi_\varepsilon \to \psi$.*

Now we are going to prove the following:

Theorem 3. *(on a continuation of the H-measure.) If $|\psi_\varepsilon(x)| \le C_3$, where $x \in \Omega$, and $C_3 = \text{const}$ is a constant independent of ε, then the measure μ, associated with the (sub)sequence $\psi_\varepsilon(x) - \psi(x)$, has a natural continuation onto $L_2(\Omega, C(S^1))$.*

The proof of Theorem 3 is based on the following auxiliary assertion, which is a direct consequence of the definition of H-measure and some properties of the Lebesgue measure on Ω (and we state it without a proof):

Lemma 2. *The measure μ is absolutely continuous with respect to the Lebesgue measure on Ω.*

Proof of Theorem 3. *i)* In the strength of Lemma 2 and the Lebesgue – Nikodym theorem [4, Ch. 5, §5.5], there exists a representation of the H-measure in the form

$$\langle \mu,\ f \rangle = \int_\Omega dx \int_{S^1} f(x, y)d\nu_x(y), \quad \text{(i.e.} \quad d\mu(x, y) = dxd\nu_x(y)),$$

where $x \to \nu_x$ is a weakly measurable with respect to the Lebesgue measure on Ω mapping of Ω into a space of Borelian measures on S^1.

In the rest of the proof, we investigate the properties of the mapping $x \to \nu_x$.

ii) The bound for the norm of ν_x. Let $f = f(x)$, $f \in C_0(\Omega)$. In the strength of the hypothesis of Theorem 3 and in the strength of formula (34), one has

$$|\langle \mu,\ f \rangle| = |\int_\Omega f(x)\|\nu_x(\cdot)\|dx| \le \|f\|_{2,\Omega} \sup_\varepsilon \|\psi_\varepsilon - \psi\|_{2,\Omega} \sup_\varepsilon \|\psi_\varepsilon - \psi\|_{\infty,\Omega} \le C_4\|f\|_{2,\Omega},$$

where $C_4 = 8(\text{ meas }\Omega)^2\, C_3$.

Since $C_0(\Omega)$ is dense in $L_2(\Omega)$, then, for any function $\varphi \in L_2(\Omega)$, the integral $\int_\Omega \varphi(x)\|\nu_x(\cdot)\|dx$ defines a linear continuous functional on $L_2(\Omega)$. In the strength of the Riesz theorem on representations, it is valid that $\|\nu_x(\cdot)\| \in L_2(\Omega)$ and $\|\|\nu_x(\cdot)\|\|_{2,\Omega} \le C_4$.

iii) Thus, it is possible to define $\langle \mu,\, a\varphi\rangle = \int_\Omega dx\varphi(x)\int_{S^1} a(y)d\nu_x(y)$ for all $\varphi \in L_2(\Omega)$ and $a \in C(S^1)$. It remains to notice that the linear span of the set $\{\varphi(x)\ a(y)\ \mid\ \varphi(x) \in L_2(\Omega),\ \ a(y) \in C(S^1)\}$ is dense in $L_2(\Omega,\ C(S^1))$, consequently, the duality brackets $\langle \mu,\, f\rangle = \int_\Omega dx \int_{S^1} f(x,y)d\nu_x(y)$ make sense for all $f \in L_2(\Omega, C(S^1))$.

Theorem 3 is proved. □

3 Transport properties of H-measures

3.1 A Theorem on transport properties of H-measures (formulation)

Consider the couple $\rho_\varepsilon(x,t)$ and $\vec{v}_\varepsilon(x,t)$ and the couple $\rho(x,t)$ and $\vec{v}(x,t)$, which are solutions of Problem A corresponding to initial data $\rho_{0\varepsilon}(x)$, $\vec{v}_{0\varepsilon}(x)$, and $\rho_0(x)$, $\vec{v}_0(x)$, respectively. In the strength of Theorems 1 and 2, the (sub)sequence $\{\rho_\varepsilon - \rho\}$ generates the H-measure $\mu(t,x,y)$, which is defined for a.e. $t \in (0,T]$ and depends on t as on a parameter, and the (sub)sequence $\{\rho_{0\varepsilon} - \rho_0\}$ generates the H-measure μ_0.

Keeping track of the proof of Theorem 3, one can establish that the norm of the measure $\mu(t,x,y)$ admits the bound

$$\|\|\mu(t,\cdot,\cdot)\|\|_{2,[0,T]} \le \ \text{const}\ \sup_\varepsilon \|\rho_\varepsilon - \rho\|_{2,Q_T} \sup_\varepsilon \|\rho_\varepsilon - \rho\|_{\infty,Q_T}.$$

This bound together with Lemma 2 and the Lebesgue – Nikodym theorem immediately implies the following:

Lemma 3. *The composition of the Lebesgue measure on $[0,T]$ and the H-measure $\mu(t,x,y)$ makes sense, and for any function $f \in L_2(Q_T, C(S^1))$ the following representation is valid.*

$$\int_0^T dt \int_{\Omega\times S^1} f(t,x,y)d\mu(t,x,y) = \int_0^T dt \int_\Omega dx \int_{S^1} f(t,x,y)d\nu_{t,x}(y),$$

$$\text{(that is } dt\, d\mu(t,x,y) = dt\, dx\, d\nu_{t,x}(y)).$$

Here, $(t,x) \to \nu_{t,x}$ is a weakly measurable with respect to the Lebesgue measure on $[0,T]\times\Omega$ mapping of $[0,T]\times\Omega$ onto S^1. For the norm of the measure $\nu_{t,x}$, the bound $\|\|\nu_{t,x}(\cdot)\|\|_{2,Q_T} \le C_5$ takes place, where $C_5 = C_5(T,\Omega,C_3)$.

We parametrize the unit circle by means of the angular coordinate y: $S^1 = \{y(\text{ mod } 2\pi)\}$.

The main result of the present section is the following theorem.

Theorem 4. *(on transport properties of H-measures). The H-measure $\mu(t,x,y)$, $(t,x,y) \in [0,T]\times\Omega\times S^1$ associated with a (sub)sequence $\{\rho_\varepsilon - \rho\}$, is a solution to the following Cauchy problem for the linear transport equation:*

$$\begin{cases} D_t\mu + \ div_x(\mu\,\vec{v}) + \frac{\partial}{\partial y}(\mu\, Y\ :\nabla_x\vec{v}) = 0, & (t,x,y) \in [0,T]\times\Omega\times S^1, \\ \mu(0,x,y) = \mu_0(x,y), \quad (x,y) \in \Omega\times S^1, \end{cases} \tag{35}$$

where $Y = \begin{pmatrix} -\frac{1}{2}\sin 2y & \cos^2 y \\ -\sin^2 y & \frac{1}{2}\sin 2y \end{pmatrix}$.

A solution of the problem (35) is understood in the sense of the integral equality

$$\int_0^T dt \int_{\Omega\times S^1} (D_t\Phi + \vec{v}\nabla_x\Phi + (Y\ :\nabla_x\vec{v})\ \frac{\partial}{\partial y}\Phi)d\mu(t,x,y) + \int_{\Omega\times S^1} \Phi(x,y,0)d\mu_0(x,y) = 0, \quad (36)$$

where $\Phi(t,x,y) \in C^1([0,T]\times\Omega\times S^1)$ is a test function satisfying the conditions $\Phi|_{\partial\Omega} = 0$, $\Phi|_{t=T} = 0$. It is worth to notice that the expression under the integral sign with respect to t is in $L_1([0,T])$. Thus, equality (36) makes sense in view of Lemma 4.

3.2 A commutator of a multiplier and a p.d.o. of zero order

In this subsection, we are going to establish some preliminary results that will be helpful for verification of Theorem 4.

Let A be a pseudo-differential operator of zero order with a principal symbol $a(x') \in C^1(S^1)$. According to [5], the inverse Fourier transform of the function $a(\frac{\xi}{|\xi|})$ is a singular kernel $K(x)$ of the form $\omega(x)/|x|^2$, where $\omega \in C^1(R^2\setminus\{0\})$ is a homogeneous function of zero order, $\int_{S^1}\omega(x')d\sigma(x') = 0$, ($x' \in S^1$ and σ is the induced Euclidean measure on S^1), and operator A has the representation in the form $A[\psi] = K*\psi$, $\psi \in L_2(R^2)$. Let B be the operator of the multiplication by a function $b \in W_2^2(\Omega)$. Evidently, B is a continuous operator defined in $L_p(\Omega)$ for any p, since $W_2^2(\Omega) \subset L_\infty(\Omega)$.

Lemma 4. *The commutator C of the operators A and B ($C \stackrel{def}{=} AB - BA$) is a continuous operator mapping from $L_\infty(\Omega)$ into $W_1^1(\Omega)$, and the following bound holds true*

$$\|D_jC[h]\|_{1,\Omega} \le C_6(\|a\|_{C^1(S^1)} + \|\omega\|_{C^1(S^1)})\|b\|_{W_2^2(\Omega)}\|h\|_{\infty,\Omega} \quad (j=1,2), \quad (37)$$

where $h \in L_\infty(\Omega)$ is an arbitrary function, and $C_6 = C_6(\Omega)$ is a constant.

Proof. Let $\{b^\nu(x)\} \subset C^\infty(\Omega)$, $b^\nu \to b$ in $W_2^2(\Omega)$. Denote by $C^\nu = AB^\nu - B^\nu A$ the commutator in which the function $b^\nu(x)$ is on the place of b. In the strength of [2], C^ν is a continuous operator mapping from $L_2(\Omega)$ into $W_2^1(\Omega)$ (hence, also, from $L_\infty(\Omega)$ into $W_1^1(\Omega)$).

Let $w \in L_\infty(\Omega)$. Consider the integral

$$I = \int_\Omega D_jC^\nu[h](x)w(x)dx \equiv \int_\Omega w(x)D_j\int_\Omega \frac{\omega(x-z)}{|x-z|^2}(b^\nu(z)-b^\nu(x))h(z)dzdx. \quad (38)$$

In the strength of the properties of the Fourier transform, one has

$$\begin{aligned} F[D_jC^\nu[h]](\xi) &= 2i\pi\xi_j\int_{R^2}\Big(a(\xi/|\xi|) - a(\eta/|\eta|)\Big)F[b^\nu](\xi-\eta)F[h](\eta)d\eta \\ &= \int_{R^2}[2i\pi\xi_j a(\xi/|\xi|) - 2i\pi\eta_j a(\eta/|\eta|)]\,F[b^\nu](\xi-\eta)F[h](\eta)d\eta \\ &\quad - \int_{R^2} a(\eta/|\eta|)F[D_jb^\nu](\xi-\eta)F[h](\eta)d\eta = F[I_1(x)+I_2(x)](\xi), \quad (39)\end{aligned}$$

where

$$I_1(x) = \int_\Omega \frac{\partial}{\partial(x_j-z_j)}\left[\frac{\omega(x-z)}{|x-z|^2}\right](b^\nu(z)-b^\nu(x))h(z)dz,$$

$$I_2(x) = D_jb^\nu(x)\int_\Omega \frac{\omega(x-z)}{|x-z|^2}h(z)dz \equiv D_jb^\nu(x)A[h](x).$$

In view of (38) and (39), we conclude that

$$I = \int_\Omega w(x) \int_\Omega \frac{\partial}{\partial(x_j - z_j)} \left[\frac{\omega(x-z)}{|x-z|^2} \right] (b^\nu(z) - b^\nu(x)) h(z) dz dx - \int_\Omega w(x) D_j b^\nu(x) A[h](x) dx. \tag{40}$$

Consider the Taylor representation

$$b^\nu(z) - b^\nu(x) = -\sum_{k=1}^{2} D_k b^\nu(z)(z_k - x_k)$$
$$-\sum_{k,l=1}^{2} \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 D_k D_l b^\nu(z + \vec{e}_l \lambda_1 \lambda_2 \delta_{kl}(z_k - x_k) + \vec{e}_l \lambda_2 (z_l - x_l))(z_k - x_k)(z_l - x_l), \tag{41}$$

where $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$ are the basis vectors in R^2 and δ_{kl} $(k,l = 1,2)$ is the Kronecker symbol.

Substituting (41) into (40), we derive

$$I \equiv I_a + I_b + I_c = \sum_{k=1}^{2} \int_\Omega w(x) \int_\Omega \frac{\partial}{\partial(x_j - z_j)} \left[\frac{\omega(x-z)}{|x-z|^2} \right] (x_k - z_k)(D_k b^\nu(z)) h(z) dz dx$$
$$- \sum_{k,l=1}^{2} \int_{\Omega\times\Omega} w(x) \frac{\partial}{\partial(x_j - z_j)} \left[\frac{\omega(x-z)}{|x-z|^2} \right] (x_k - z_k)(x_l - z_l) h(z)$$
$$\times \left\{ \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 D_k D_l b^\nu(z + \vec{e}_l \lambda_1 \lambda_2 \delta_{kl}(z_k - x_k) + \vec{e}_l \lambda_2 (z_l - x_l)) \right\} dz dx$$
$$- \int_\Omega w(x)(D_j b^\nu(x)) A[h](x) dx. \tag{42}$$

Let us estimate each of the integrals I_a, I_b and I_c:

$$I_a = \sum_{k=1}^{2} \int_\Omega w(x) \tilde{A}_{jk}[(D_k b^\nu) h](x) dx,$$

where

$$\tilde{A}_{jk}[D_k b^\nu h](x) = \int_\Omega \frac{\partial}{\partial(x_j - z_j)} \left[\frac{\omega(x-z)}{|x-z|^2} \right] (x_k - z_k)(D_k b^\nu(z)) h(z) dz.$$

In view of the properties of the Fourier transform, the identity

$$F[\tilde{A}^{ij}[(D_k b^\nu) h](\xi) = \frac{\partial}{\partial \xi_j} a(\frac{\xi}{|\xi|}) \xi_i \; F[(D_k b^\nu) h](\xi),$$

holds true. Here we have the p.d.o. of zero order $\tilde{A}^{ij}$ with the principal symbol $\frac{\partial}{\partial \xi_j} a(\frac{\xi}{|\xi|})\xi_i$. Its norm evidently admits the bound

$$\|\tilde{A}^{ij}[h](t)\|_2 \le C_7 \|a\|_{C^1(S^1)} \|D_k b^\nu h\|_{2,\Omega}.$$

Here, $C_7 = C_7(\Omega)$ is a constant not depending on h and a. In its turn, this bound yields the following one:

$$|I_a| \le C_8(\Omega) \|w(x)\|_{\infty,\Omega} \|a\|_{C^1(S^1)} \|b_x^\nu\|_{2,\Omega} \|h\|_{\infty,\Omega} \tag{43}$$

Let us change variable under the integral sign in I_b: $x = q + z$. We obtain

$$I_b = -\sum_{k,l=1}^{2} \int_{B\times\Omega} w(q+z)\frac{\partial}{\partial q_j}\left[\frac{\omega(q)}{|q|^2}\right] q_k q_l h(z)$$
$$\times \left\{\int_0^1 d\lambda_1 \int_0^1 d\lambda_2 D_k D_l b^\nu(z + \vec{e}_l\lambda_1\lambda_2\delta_{kl}q_k + \vec{e}_l\lambda_2 q_l)\right\} dqdz.$$

Here B is the domain, which contains the supporter of the function $w(q+z)$, $z \in \Omega$, $x \in \Omega$. One has $\operatorname{meas} B \le 4\pi(\operatorname{diam}\Omega)^2$, where $\operatorname{diam}\Omega$ is the diameter of the smallest ball containing Ω.

According to Holder's inequality, we have

$$|I_b| \le \sum_{k,l=1}^{2} \|h\|_{\infty,\Omega}\|w\|_{\infty,\Omega}\Big(\int_B |\frac{\partial}{\partial q_j}\left[\frac{\omega(q)}{|q|^2}\right] q_k q_l|dq\Big)\|D_k D_l b^\nu\|_{1,\Omega}.$$

In view of the bound $\|D_k D_l b^\nu\|_{1,\Omega} \le C_9(\Omega)\|D_k D_l b^\nu\|_{2,\Omega}$ we have

$$\int_B |\frac{\partial}{\partial q_j}\left[\frac{\omega(q)}{|q|^2}\right] q_k q_l|dq \le \|\omega\|_{C^1(S^1)}\int_B \frac{dq}{|q|} \le C_{10}(\Omega)\|\omega\|_{C^1(S^1)}.$$

Finally, we get the following estimate for I_b:

$$|I_b| \le C_{11}(\Omega)\|h\|_{\infty,\Omega}\|w\|_{\infty,\Omega}\|\omega\|_{C^1(S^1)}\|b^\nu_{xx}\|_{2,\Omega}. \tag{44}$$

According to the properties of the p.d.o. A and to Holder's inequality, the following estimate is valid:

$$|I_c| \le \|w\|_{\infty,\Omega}\|D_j b^\nu\|_{2,\Omega}\|a\|_{C(S^1)}\|h\|_{2,\Omega} \le C_{12}(\Omega)\|w\|_{\infty,\Omega}\|b^\nu_x\|_{2,\Omega}\|a\|_{C^1(S^1)}\|h\|_{\infty,\Omega}. \tag{45}$$

Aggregating (43)–(45), we obtain

$$\|D_j C^\nu[h]\|_{1,\Omega} \le C_6(\|a\|_{C^1(S^1)} + \|\omega\|_{C^1(S^1)})\|b^\nu\|_{W^2_2(\Omega)}\|h\|_{\infty,\Omega}, \tag{46}$$

where $C_6 = \max\{C_8 + C_{12}, C_{11}\}$. Since $b^\nu \to b$ in $W^2_2(\Omega)$, one has that C is a continuous operator mapping from L_∞ into $W^1_1(\Omega)$. Besides, the bound (37) is valid. Lemma 4 is proved. □

3.3 Proof of Theorem 4

Let $\rho^\nu = \rho * \omega_\nu$ be the regularization of the density ρ by means of usual mollifying kernel ω_ν. Let A be an arbitrary p.d.o. of zero order with symbol $a \in C^1(S^1)$, and let w be an arbitrary function such that $w \in C^1(Q_T)$, $w|_{\partial\Omega} = 0$ and $w|_{t=T} = 0$. Substituting $\varphi = A[\rho^\nu]w$ on the place of the test function into (8) we derive

$$\int_{Q_T} \rho(D_t(A[\rho^\nu]w) + \sum_{i=1}^{2} v_i D_i(A[\rho^\nu]w)dxdt + \int_\Omega \rho_0 A[\rho^\nu(0)]w(x,0)dx = 0 \tag{47}$$

Operator A is self-adjoint in L_2 and commutes with D_i (i=1,2). Thus, (47) can be transformed to the form

$$\int_{Q_T} \rho\left[A[\rho^\nu]D_t w + \sum_{i=1}^{2} v_i A[\rho^\nu]D_i w\right] dxdt + \int_{Q_T} A[\rho w]\left(D_t\rho^\nu + \sum_{i=1}^{2} v_i D_i\rho^\nu\right) dxdt$$
$$- \int_{Q_T} \rho^\nu \sum_{i=1}^{2} D_i\{A[v_i w] - v_i A[w]\}dxdt + \int_\Omega \rho_0 A[\rho^\nu(0)]w(x,0)dx = 0 \tag{48}$$

In the strength of [6, lemma II.1 and corollary II.2] we have

$$D_t\rho^\nu + \sum_{i=1}^{2} v_i D_i \rho^\nu \to 0 \text{ in } L_2(Q_T), \quad \rho^\nu(x,0) \to \rho_0 \text{ in } C([0,T]; L_p(\Omega))\ \forall\, p<\infty \tag{49}$$

as $\nu \to 0$. In the strength of Lemma 4 and the properties of the p.d.o. A, one has

$$A[\rho]\, D_t w \,+\, A[\rho]\, \vec v\, \nabla_x w \in L_2(Q_T), \quad \mathrm{div}_x(A[\vec v\rho] - \vec v A[\rho]) \in L_1(Q_T), \quad A[\rho_0]\, w(x,0) \in L_2(\Omega).$$

Thus, it is possible to fulfill the limiting transition in (48) as $\nu \to 0$, and to obtain, as the result, the following:

$$\int_{Q_T} \rho\Big(A[\rho]\, D_t w \,+\, A[\rho]\, \vec v\, \nabla_x w\Big) dxdt \\ - \int_{Q_T} \mathrm{div}_x(A[\vec v\rho] - \vec v A[\rho])\rho\, w\, dxdt + \int_\Omega \rho_0 A[\rho_0]\, w(x,0)\, dx = 0. \tag{50}$$

Considering in the same way Problem A with the initial data $\rho_{0\varepsilon}$ and $\vec v_{0\varepsilon}$, we obtain

$$\int_{Q_T} \rho_\varepsilon\Big(A[\rho_\varepsilon]\, D_t w \,+ A[\rho_\varepsilon]\, \vec v_\varepsilon\, \nabla_x w\Big) dxdt \\ - \int_{Q_T} \mathrm{div}_x(A[\vec v_\varepsilon\rho_\varepsilon] - \vec v_\varepsilon A[\rho_\varepsilon])\rho_\varepsilon\, w\, dxdt + \int_\Omega \rho_{0\varepsilon} A[\rho_{0\varepsilon}]\, w(x,0)\, dx = 0. \tag{51}$$

Since the set $\{\vec\phi \mid \vec\phi \in C_0^1(Q_T),\ \mathrm{div}_x\vec\phi = 0\}$ is dense in $L_2(0,T;H^1(\Omega))$, it is possible to construct the sequence $\{\vec v^{(n)}\}$ such that

$$\mathrm{supp}\ \vec v^{(n)} \subset\ Q_T,\ n=1,2,\dots,\ \text{ and } \vec v^{(n)} \to \vec v \text{ in } L_2(0,T;\ H^1(\Omega)). \tag{52}$$

Introducing into (50) the sum $\vec v^{(n)} + (\vec v - \vec v^{(n)})$ on the place of $\vec v$ and aggregating the result with identity (51), we derive

$$\int_{Q_T} (\rho_\varepsilon - \rho)\ A[\rho_\varepsilon - \rho]\ (D_t w \;+\; \vec v^{(n)}\ \nabla_x w) dxdt \\ - \int_{Q_T} \mathrm{div}_x(A[\vec v^{(n)}(\rho_\varepsilon - \rho)] - \vec v^{(n)} A[\rho_\varepsilon - \rho])(\rho_\varepsilon - \rho)\ w\ dxdt \\ + \int_\Omega (\rho_{0\varepsilon} - \rho_0)\ A[\rho_{0\varepsilon} - \rho_0]\ w(x,0)\ dx \\ + \Big\{\int_{Q_T} (\rho\ A[\rho_\varepsilon] \;+\; \rho_\varepsilon A[\rho])(D_t w \;+ \vec v^{(n)} \nabla_x w) dxdt \;+ \int_\Omega (\rho_{0\varepsilon}\ A[\rho_0] + \rho_0 A[\rho_{0\varepsilon}])\ w(x,0)\ dx \\ - \int_{Q_T} \mathrm{div}_x(A[\vec v^{(n)}\rho_\varepsilon] - \vec v^{(n)} A[\rho_\varepsilon])\rho\ w\ dxdt \;- \int_{Q_T} \mathrm{div}_x(A[\vec v^{(n)}\rho] - \vec v^{(n)} A[\rho])\rho_\varepsilon\ w\ dxdt\Big\} \\ + \Big[\int_{Q_T} \rho_\varepsilon\ A[\rho_\varepsilon]\ (\vec v_\varepsilon - \vec v^{(n)})\ \nabla_x w) dxdt \;+\; \int_{Q_T} \rho\ A[\rho]\ (\vec v - \vec v^{(n)})\ \nabla_x w) dxdt \\ - \int_{Q_T} \mathrm{div}_x(A[(\vec v_\varepsilon - \vec v^{(n)})\rho_\varepsilon] - (\vec v_\varepsilon - \vec v^{(n)}) A[\rho_\varepsilon])\rho_\varepsilon\ w\ dxdt \\ - \int_{Q_T} \mathrm{div}_x(A[(\vec v - \vec v^{(n)})\rho] - (\vec v - \vec v^{(n)}) A[\rho])\rho\ w\ dxdt\Big] = 0. \tag{53}$$

3.3.1 Limiting transitions as $\varepsilon \to 0$ and $n \to \infty$

Repeating the considerations from [2, pages 14–16], we obtain that the first three summands in the left hand side of (53) tend to the following expression, as $\varepsilon \to 0$:

$$I^{(n)} \equiv \int_0^T \langle \mu,\, a\, D_t w\rangle\, dt + \int_0^T \langle \mu,\, a\vec{v}^{(n)} \cdot \nabla_x w\rangle dt - \int_0^T \langle \mu,\, \{a,\ v_1^{(n)}\xi_1 + v_2^{(n)}\xi_2\}\ w\rangle\, dt + \langle \mu_0,\, a\, w(0,x)\rangle, \tag{54}$$

where $\xi \in R^2$, and the Poisson bracket is $\{\alpha,\beta\} = \sum_{j=1}^2 \frac{\partial\alpha}{\partial x_j}\frac{\partial\beta}{\partial\xi_j} - \frac{\partial\beta}{\partial x_j}\frac{\partial\alpha}{\partial\xi_j}$. We remark that differentiating a test function Φ with respect to ξ_i ($i = 1, 2$) does not output integrands off the domain of definition of measure μ_t because the Poisson bracket in (54) is continuous in $[0,T] \times \Omega \times S^1$ and homogeneous of zero order with respect to the variable $\vec{\xi}$.

We parametrize the unit circle S^1 by means of the radial and angular coordinates (r, y): $\xi_1 = r\cos y$, and $\xi_2 = r\sin y$, and change variables in (54). Eventually, after simple computations, we obtain the following:

$$I^{(n)} = \int_0^T \langle \mu,\, \tilde{a} D_t w\rangle\, dt + \int_0^T \langle \mu,\, \tilde{a}\vec{v}^{(n)} \cdot \nabla_x w\rangle\, dt + \int_0^T \langle \mu,\, (Y : \nabla_x \vec{v}^{(n)})\frac{\partial\tilde{a}}{\partial y}\, w\rangle\, dt + \langle \mu_0, \tilde{a}\, w(0,x)\rangle, \tag{55}$$

where $Y = \begin{pmatrix} -\frac{1}{2}\sin 2y & \cos^2 y \\ -\sin^2 y & \frac{1}{2}\sin 2y \end{pmatrix}$ and we denote $\tilde{a}(y) \stackrel{def}{=} a(\cos y, \sin y)$ $(= a(\xi/|\xi|))$, ($\tilde{a}(y)$ is 2π-periodic function).

In the strength of Theorem 2.2 and Lemma 3.1, we obtain from (55) that

$$\lim_{n\to\infty} I^{(n)} = \int_0^T \langle \mu,\, \tilde{a}\, D_t w\rangle\, dt + \int_0^T \langle \mu,\, \tilde{a}\vec{v} \cdot \nabla_x w\rangle\, dt + \int_0^T \langle \mu,\, (Y : \nabla_x \vec{v})\frac{\partial\tilde{a}}{\partial y}\, w\rangle\, dt + \langle \mu_0,\, \tilde{a}\, w(0,x)\rangle. \tag{56}$$

Now, let us show that the other terms in the left hand side of(53) tend to zero as $\varepsilon \to 0$ and $n \to \infty$. Firstly, we consider the expression in parenthesis $\{\ldots\}$. In the strength of Theorem 1.1, formula (52) and continuity of the operator A in L_2, the following limiting relations hold true.

$$\int_{Q_T} (\rho\, A[\rho_\varepsilon] + \rho_\varepsilon A[\rho])(D_t w + \vec{v}^{(n)}\nabla_x w) dxdt \to 2\int_{Q_T} \rho\, A[\rho](D_t w + \vec{v}\nabla_x w) dxdt, \tag{57}$$

$$\int_\Omega (\rho_{0\varepsilon}\, A[\rho_0] + \rho_0 A[\rho_{0\varepsilon}])\, w(x,0)\, dx \to 2\int_\Omega \rho_0\, A[\rho_0]\, w(x,0)\, dx. \tag{58}$$

Passing to the limit as ε and n tend to zero in the last two terms in parenthesis in (53) and applying Lemma 3.2, we derive that

$$\int_{Q_T} \operatorname{div}_x(A[\vec{v}^{(n)}\rho_\varepsilon] - \vec{v}^{(n)} A[\rho_\varepsilon])\rho\, w\, dxdt \to \int_{Q_T} \operatorname{div}_x(A[\vec{v}\rho] - \vec{v}A[\rho])\rho\, w\, dxdt, \tag{59}$$

$$\int_{Q_T} \operatorname{div}_x(A[\vec{v}^{(n)}\rho] - \vec{v}^{(n)} A[\rho])\rho_\varepsilon\, w\, dxdt \to \int_{Q_T} \operatorname{div}_x(A[\vec{v}\rho] - \vec{v}A[\rho])\rho\, w\, dxdt. \tag{60}$$

Formulae (57)–(60) yield that the limit of the expression in the parenthesis in (53) is the doubled left hand side of (50). Thus, this expression is equal to zero.

Finally, consider four integrals in the square brackets in (53). Every of these integrals tends to zero in the strength of the bound (37) and the limiting relations $\vec{v}_\varepsilon \to \vec{v}$ and $\vec{v}^{(n)} \to \vec{v}$.

Thus, after the limiting transition in (53), we derive

$$\int_0^T \langle \mu,\, \tilde{a}\, D_t w\rangle\, dt + \int_0^T \langle \mu,\, \tilde{a}\vec{v}\cdot\nabla_x w\rangle\, dt + \int_0^T \langle \mu,\, (Y\, :\, \nabla_x \vec{v})\frac{\partial \tilde{a}}{\partial y}\, w\rangle\, dt + \langle \mu_0,\, \tilde{a}\, w(0,x)\rangle = 0. \quad (61)$$

Note that the linear span of the set $\{\tilde{a}w \mid \tilde{a} \in C^1(S^1),\ \ w \in C^1(Q_T),\ \ w|_{\partial\Omega} = 0,\ \ w|_{t=T} = 0\}$ is dense in $\{\Phi(t,x,y) | \, \Phi \in C^1(Q_T \times S^1),\ \Phi|_{\partial\Omega} = 0,\ \Phi|_{t=T} = 0\}$. Consequently, equality (61) yields (36). Theorem is proved. □

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