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Generalized Lagrangian Coordinates and the Uniqueness of the Solution of a Linear Transport Equation

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1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

We consider the Cauchy problem for the linear transport equation

$$\partial_t u + \vec{v} \cdot \nabla_x u + cu = 0, \qquad u|_{t=0} = u_0, \qquad (\vec{x}, t) \in Q_T, \tag{1.1}$$

where $Q_T = \Omega \times (0, T)$, $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary, $0 < T < +\infty$, $\vec{v} : Q_T \to \mathbf{R}^N$ and $c, u_0 : Q_T \to \mathbf{R}$ are known functions, $u : Q_T \to \mathbf{R}$ is the unknown function, and the following notation is used: $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, and $\nabla_x = \{\partial_1, \ldots, \partial_N\}$.

This equation is used in numerous models of mechanics; hence the interest in well-posedness issues for problem (1.1) in various function spaces. In Lebesgue classes, the problem of finding minimal smoothness conditions for the coefficients $\vec{v} = \{v_1, \ldots, v_N\}$ (which are treated as the velocity vector in mechanics) and the coefficient c providing the unique solvability of problem (1.1) was studied for the first time in [1], where it was shown that there exists a unique solution $u \in L_\infty(0, T; L_p(\Omega))$ provided that $u_0 \in L_p(\Omega)$, $\vec{v} \in L_1\left(0, T; W_q^1(\Omega)\right)$ $(p^{-1} + q^{-1} = 1)$, and $\operatorname{div}_x \vec{v}, c \in L_1(0, T; L_\infty(\Omega))$, where $\operatorname{div}_x \vec{v} = \partial_1 v_1 + \cdots + \partial_N v_N$.

The smoothness assumptions about the divergence $\operatorname{div}_x \vec{v}$ of the velocity vector and the coefficient c for which there exists a unique solution $u \in L_\infty(Q_T)$ were weakened in [2, 3] for the case in which $u_0 \in L_\infty(\Omega)$. It was shown in [2] that it suffices to require the existence of a positive constant C_0 such that $\exp(C_0 |\operatorname{div}_x \vec{v}|) \in L_1(Q_T)$ and $\exp(C_0|c|) \in L_1(Q_T)$, and this requirement was replaced in [3] by the less restrictive condition that $\operatorname{div}_x \vec{v} \in K_M(Q_T)$ and $c \in K_M(Q_T)$, where $M \in \mathscr{K}$, $\mathscr{K}$ is the set of even functions $M(s) \geq 0$ such that the antiderivative of the product $s^{-2} \ln M(s)$ grows unboundedly as $s \to +\infty$ and $K_M(Q_T)$ is the Orlicz class generated by M: $K_M(Q_T) = \left\{\varphi \,\middle|\, \int_{Q_T} M(\varphi(\vec{x}, t))\, d\vec{x}\, dt < \infty\right\}$.

The main result of the present paper is the following theorem on the uniqueness of the solution of problem (1.1) in the case of a solenoidal velocity field ($\operatorname{div}_x \vec{v} = 0$).

Theorem 1.1 (on the uniqueness of the solution of the transport equation). *Let*

$$\vec{v} \in L_\gamma\left(0, T; \mathring{W}_\alpha^1(\Omega) \cap J(\Omega)\right), \qquad u_0 \in L_p(\Omega), \qquad c \in L_\gamma(0, T; L_q(\Omega)), \tag{1.2}$$

where $1 < \gamma < \infty$ and the exponents α, p, and q satisfy the following conditions: if $N > \alpha$, then $q \geq \alpha N(N-\alpha)^{-1}$ and $p^{-1} \leq 1 - (N-\alpha)(\alpha N)^{-1}$; if $N \leq \alpha$, then $q^{-1} \leq 1 - p^{-1}$ and $p > 1$ can be arbitrary. Then problem (1.1) *with the initial data u_0 has at most one solution $u \in L_\delta(0, T; L_p(\Omega))$, $\delta^{-1} + \gamma^{-1} \leq 1$.*

Here and in the following, $J(\Omega)$ stands for the closure of the space of infinitely differentiable solenoidal compactly supported vector fields in the norm of $L_2(\Omega)$.

The main difficulty in the proof of Theorem 1.1 is as follows: since the norm $\|c\|_{L_1(0,T;L_\infty(\Omega))}$ is unbounded, one cannot use the well-known technique [1–3] in which estimates of solutions are obtained with the aid of the Gronwall lemma or its generalizations [3]. The idea of the proof of Theorem 1.1 is to use the Lagrangian representation of Eq. (1.1); then the equation is simplified, which permits one to avoid the above-mentioned difficulties. However, we encounter another difficulty: $\vec{v} \in L_1(0,T;W_2^2(\Omega))$ ($\Omega \subset \mathbf{R}^3$) is the minimal smoothness condition on the velocity field $\vec{v}(\vec{x},t)$ under which the possibility of replacing Eulerian variables by Lagrangian variables in functions of the class $L_p(\Omega)$ has been studied and justified in [4], and this problem remains open in the case of lower smoothness of the velocity field. In the present paper, we suggest the notion of a Lagrangian transformation, which is a generalization of the Lagrangian representation to the case of solenoidal velocity fields from the space $L_\gamma(0,T;W_\alpha^1(\Omega))$.

A Lagrangian transformation is given by the Lagrange operator defined as follows. For any given $t \in [0,T]$ and for any function $f(\cdot,t) \in L_p(\Omega)$, we define $F^{(t)}(\vec{x},t)$ as the solution of the Cauchy problem

$$Q_T: \ \partial_s F^{(t)} + \operatorname{div}_x\left(\vec{v}(\vec{x},s)\,F^{(t)}\right) = 0, \qquad \Omega: \ \ F^{(t)}(\vec{x},s)\big|_{s=t} = f(\vec{x},t).$$

If $f \in L_\vartheta(0,T;L_p(\Omega))$, then, for almost every $t \in [0,T]$, there exists a unique solution $F^{(t)}(\vec{x},s)$, which, treated as a function of the variables $\vec{x}$ and s, belongs to the space $C([0,T];L_p(\Omega))$ (if $p < \infty$) or $L_\infty(Q_T) \cap C([0,T];L_{p'}(\Omega))$ with arbitrary $p' < \infty$ (if $p = \infty$) [1, Corollaries II.1, II.2].

Definition 1.1. The Lagrange operator $\mathscr{L} : L_p(\Omega) \to L_p(\Omega)$ corresponding to the vector field $\vec{v}$ is defined by the formula $\mathscr{L}[f](\vec{x},t) = F^{(t)}(\vec{x},0)$, $t \in [0,T]$. The image of a function f under the mapping $\mathscr{L}$ is referred to as the *Lagrange transform* of this function.

Proposition 1.1 (on properties of the operator $\mathscr{L}$). *If*

$$f \in L_\vartheta(0,T;L_p(\Omega)), \qquad \vec{v} \in L_\gamma\left(0,T;\overset{\circ}{W}{}_\alpha^1(\Omega) \cap J(\Omega)\right),$$

then the following assertion are valid:

(1) $\mathscr{L}[f](\vec{x},t)$ *is a measurable function in* Q_T; *moreover,*

(2) $\mathscr{L}[f] \in L_\vartheta(0,T;L_p(\Omega))$;

(3) $\|\mathscr{L}[f]\|_{L_\vartheta(0,T;L_p(\Omega))} \le \|f\|_{L_\vartheta(0,T;L_p(\Omega))}$, *where the equality takes place for* $p < \infty$;

(4) *if* $\vec{v} \in C^1([0,T];C_0^1(\Omega) \cap J(\Omega))$ *and the mapping* $f : Q_T \to \mathbf{R}$, $f \in C^1(Q_T)$, *is defined in the Eulerian coordinates, then* $\mathscr{L}[f](\vec{\xi},t) = [f]_\xi(\vec{\xi},t)$, *where* $[f]_\xi$ *is the representation of the function* f *in the Lagrangian coordinates*;

(5) *there exists an inverse* $\mathscr{L}^{-1}$ *of the Lagrange operator* $\mathscr{L}$, *i.e.,* $\mathscr{L} \circ \mathscr{L}^{-1}$ *and* $\mathscr{L}^{-1} \circ \mathscr{L}$ *are the identity transformation on* $L_p(\Omega)$ *for almost all* $t \in [0,T]$; *the operator* $\mathscr{L}^{-1}$ *satisfies assertions* (1)–(3) *of this proposition*;

(6) *if* $\vec{v} \in C^1([0,T];C_0^1(\Omega) \cap J(\Omega))$ *and the mapping* $[f]_\xi : Q_T \to \mathbf{R}$, $[f]_\xi \in C^1(Q_T)$, *is defined in the Lagrangian coordinates, then* $\mathscr{L}^{-1}[[f]_\xi](\vec{x},t) = f(\vec{x},t)$, *where* f *is the representation of the same mapping in the Eulerian coordinates.*

Here and in the following, $X \circ Y$ (respectively, $f \circ g$) stands for the composition of operators X and Y (respectively, functions f and g). The proof of Proposition 1.1 is given in Section 2. In Section 3, we prove the following assertion on the basis of Proposition 1.1.

Theorem 1.2 [on the Lagrange transformation of Eq. (1.1)]. *Suppose that* $C(\vec{x},t) = \mathscr{L}[c](\vec{x},t)$ *and the exponents* α, γ, δ, p, *and* q *satisfy the assumptions of Theorem* 1.1. *A function* $u \in L_\delta(0,T;L_p(\Omega))$ *is a solution of problem* (1.1) *if and only if* $U(\vec{x},t) = \mathscr{L}[u](\vec{x},t)$ *is the solution of the following Cauchy problem for the ordinary differential equation with the parameter* $\vec{x}$:

$$\partial_t U + CU = 0, \qquad U|_{t=0} = U_0 \equiv u_0, \qquad (\vec{x},t) \in Q_T. \tag{1.3}$$

The validity of Theorem 1.2, together with the uniqueness of the solution of Eq. (1.3), proved in Section 4, implies a similar assertion for Eq. (1.1).

2. THE LAGRANGIAN TRANSFORMATION

2.1. Preliminary Information

In the proof of Proposition 1.1 and auxiliary assertions dealing with the structure of the Lagrange operator and used in the proof of Theorem 1.2, we need the well-known properties of solutions of the Cauchy problem for the differential equation

$$\partial_t u + \vec{v}\cdot\nabla_x u = 0, \qquad u|_{t=\lambda} = u_0, \qquad \vec{x}\in\Omega, \qquad t,\lambda\in[0,T]; \tag{2.1}$$

thus, we recall some of them.

Let the vector field $\vec{v}$ and the initial function u_0 occurring in the statement of problem (2.1) be smooth; more precisely, $\vec{v}\in C^1([0,T];C_0^1(\Omega)\cap J(\Omega))$ and $u_0\in C^1(\Omega)$. Then there exists a unique classical solution $u\in C^1(Q_T)$ of problem (2.1), and this solution has the form [5, 6]

$$u(\vec{x},t) = u_0\left(\Phi^{t,\lambda}(\vec{x})\right), \tag{2.2}$$

where $\Phi^{t_1,t_2}:\Omega\to\Omega$ is the operator of shift along the trajectories, defined as

$$\Phi^{t_1,t_2}(\vec{x}) = \vec{\xi}(s)\Big|_{s=t_2},$$

where we have $(d/ds)\vec{\xi} = \vec{v}(\vec{\xi},s)$ and $\vec{\xi}|_{s=t_1} = \vec{x}$. The mapping Φ is volume-preserving (i.e., $J\equiv\det\left(\partial\Phi^{t_1,t_2}(\vec{x})/\partial\vec{x}\right) = 1$ for any $t_1,t_2\in[0,T]$ [7, Chap. II, Sec. 5, formula (II.5–8)]) and has the group property $\Phi^{t_3,t_1}\circ\Phi^{t_2,t_3} = \Phi^{t_1,t_2}$ for any $t_1,t_2,t_3\in[0,T]$.

Lemma 2.1 [on solutions of Eq. (2.1) in Lebesgue classes]. *Let*

$$\vec{v}\in L_\gamma\left(0,T;\overset{\circ}{W}{}^1_\alpha(\Omega)\cap J(\Omega)\right), \qquad u_0\in L_p(\Omega), \tag{2.3}$$

where $1\le\gamma<\infty$ *and either* $p^{-1}\le 1-(N-\alpha)(\alpha N)^{-1}$ (*if* $N>\alpha$) *or* $p\ge 1$ *is arbitrary* (*if* $N\le\alpha$). *Then the following assertions are valid*:

(i) *there exists a unique solution* $u\in L_\infty(0,T;L_p(\Omega))$ *of problem* (2.1); *moreover, if* $1\le p<\infty$, *then* $u\in C([0,T];L_p(\Omega))$ [1, *Corollaries* II.1 *and* II.2];

(ii) *either* $\|u(t)\|_{p,\Omega} = \|u_0\|_{p,\Omega}$ *for any* $t\in[0,T]$ (*if* $p\in[1,\infty)$) [8, *Chap.* III, *Sec.* 2, *Lemma* 2.1], *or* $\|u(t)\|_{\infty,\Omega}\le\|u_0\|_{\infty,\Omega}$ *for any* $t\in[0,T]$ (*if* $p=\infty$) [1, *formula* (16)];

(iii) *suppose that* $\vec{v}_n$ *and* u_{0n}, $n\ge 1$, *satisfy conditions* (2.3), $p<\infty$, $\vec{v}_n\to\vec{v}$ *in* $L_\gamma(0,T;W^1_\alpha(\Omega)\cap J(\Omega))$ *and* $u_{0n}\to u_0$ *in* $L_p(\Omega)$ *as* $n\to\infty$, $\{u_n\}$ *is the sequence of solutions of problem* (2.1) *with* $\vec{v}_n$ *and* u_{0n} *substituted for* $\vec{v}$ *and* u_0, *and the sequence* $\{u_n\}$ *is bounded in* $L_\infty(0,T;L_p(\Omega))$; *then* $u_n\to u$ *in* $C([0,T];L_p(\Omega))$, *where* u *is the solution of problem* (2.1) *with given functions* $\vec{v}$ *and* u_0 [1, *Th.* II.4];

(iv) *let* $u\in L_\infty(0,T;L_p(\Omega))$ *be a solution of problem* (2.1), *and let* $u_\varepsilon = u*\omega_\varepsilon$, *where* $\omega_\varepsilon = \varepsilon^{-1}\omega(\cdot/\varepsilon^N)$ *and* ω *is an even function from the class* $\mathscr{D}_+(\mathbf{R}^N)$ *with unit mean value*; *then* $\partial_t u_\varepsilon + \operatorname{div}_x(\vec{v}u_\varepsilon) = r_\varepsilon$, *where* $r_\varepsilon\to 0$ *strongly in* $L_\gamma(0,T;L_\beta(\Omega))$, $\beta^{-1} = \alpha^{-1}+p^{-1}$ *if either* $\alpha<\infty$ *or* $p<\infty$, *and* $\beta<\infty$ *can be arbitrary if* $\alpha = p = \infty$ [1, *Th.* II.1].

Here and in the following, $\|\cdot\|_{q,\Omega} = \|\cdot\|_{L_q(\Omega)}$ for any $q\in[1,\infty]$, and $\varphi_1*\varphi_2$ stands for the convolution of two functions: $(\varphi_1*\varphi_2)(\vec{x}) = \int_{\mathbf{R}^N}\varphi_1(\vec{x}-\vec{y})\varphi_2(\vec{y})d\vec{y}$. If a function is defined in a bounded domain Ω, then in the convolution integral it is assumed to vanish identically on $\mathbf{R}^N\backslash\Omega$.

Remark 2.1. If some sequence $\{f_n(\vec{x},t)\}$ is bounded in $L_\infty(Q_T)$ and converges to some function f in $C([0,T];L_p(\Omega))$ with an arbitrary exponent p, then $f_n\to f$ $*$-weakly in $L_\infty(Q_T)$.

This obvious fact, together with assertion (iii) of Lemma 2.1, implies the following statement.

Corollary 2.1. If the assumptions of item (iii) of Lemma 2.1 are satisfied and u_{0n} is a uniformly bounded sequence in the norm of $L_\infty(\Omega)$, then $u_n\to u$ $*$-weakly in $L_\infty(Q_T)$.

Definition 2.1. The *flow* is the vector function $\vec{X} = \vec{X}(\vec{x}, t, \lambda)$ whose components are the solutions X_i, $i = 1, \dots, N$, of problem (2.1) with the Cauchy data $X_i|_{t=\lambda} = x_i$.

The following assertion is a straightforward consequence of Definition 2.1, the representation (2.2), assertion (iii) of Lemma 2.1, and Corollary 2.1.

Lemma 2.2 (on flow properties). *Let the velocity field $\vec{v}$ occurring in problem* (2.1) *satisfy condition* (1.2). *Then the following assertions are valid*:

(i) $\vec{X} = \vec{X}(\vec{x}, t, \lambda) \in \bar{\Omega} \equiv \Omega \cup \partial\Omega$ *for arbitrary* $t, \lambda \in [0, T]$ *and for almost all* $\vec{x} \in \Omega$;

(ii) *if* $f \in C^1(\bar{\Omega})$ *and* $\vec{X}_\varepsilon$ *is the flow regularization in the sense of assertion* (iv) *of Lemma* 2.1, *then* $f(\vec{X}_\varepsilon) \to f(\vec{X})$ *in* $L_{\vartheta_1}(0, T; L_{\vartheta_2}(\Omega))$ *for arbitrary* $\vartheta_1, \vartheta_2 < \infty$ *and* $*$*-weakly in* $L_\infty(Q_T)$;

(iii) $\partial_t f(\vec{X}_\varepsilon) + \vec{v} \cdot \nabla_x f(\vec{X}_\varepsilon) \to 0$ *in* $L_\gamma(0, T; L_\beta(\Omega))$, *where* β *is defined in assertion* (iv) *of Lemma* 2.1.

2.2. Properties of the Lagrange Operator

2.1.1. Proof of Proposition 1.1. The proof of assertions (2)–(6) of Proposition 1.1 is rather simple; therefore, we only outline it. In the case of a smooth velocity field $\vec{v}$ and a smooth function f, assertion (2) of Proposition 1.1 readily follows from the existence and the properties of a classical solution of problem (2.1). Item (iii) of Lemma 2.1 permits one to generalize this assertion to the case of nonsmooth $\vec{v}$ and f. The estimates of assertion (3) of Proposition 1.1 are straightforward corollaries to assertion (ii) of Lemma 2.1. The validity of assertion (4) is based on the explicit form (2.2) of the classical solution of problem (2.1), which coincides with the representation of the function f in the Lagrangian variables $\vec{\xi}$. Finally, the inverse operator $\mathscr{L}^{-1}$ can be found for almost all $t \in [0, T]$ from the Cauchy problem

$$\partial_s R^{(t)} + \vec{v}(\vec{x}, s) \cdot \nabla_x R^{(t)} = 0, \qquad (\vec{x}, s) \in Q_T, \qquad R^{(t)}(\vec{x}, s)\big|_{s=0} = f(\vec{x}, t), \qquad \vec{x} \in \Omega,$$

by the formula $\mathscr{L}^{-1}[f](\vec{x}, t) = R^{(t)}(\vec{x}, t)$. For smooth $\vec{v}$ and f, this is an obvious consequence of the representation (2.2); the generalization to the case of nonsmooth $\vec{v}$ and f goes in accordance with assertion (iii) of Lemma 2.1. The remaining assertions (5) and (6) of Proposition 1.1 can be proved by analogy with assertions (1)–(4).

We present the proof of assertion (1) on the measurability of the transform under the mapping $\mathscr{L}$ in detail. Note that since Ω is a bounded domain, we have $L_\infty(\Omega) \subset L_r(\Omega)$ for any $r < \infty$; consequently, we can restrict out consideration to the case in which $p < \infty$ and $\alpha < \infty$. We first suppose that f belongs to the class $C^1(Q_T)$. Let

$$\{\vec{v}_n\} \subset C^1\left([0, T], C_0^1(\Omega) \cap J(\Omega)\right),$$

$\vec{v}_n \to \vec{v}$ in $L_\gamma\big(0, T; \mathring{W}_\alpha^1(\Omega)\big)$, and let $\mathscr{L}_n$ be the Lagrange operator corresponding to the vector field $\vec{v}_n$. It follows from Definition 1.1 and properties of a classical solution of Eq. (2.1) that $\mathscr{L}_n[f]$ belongs to the class $C^1(Q_T)$. By assertion (iii) of Lemma 2.1, we have the limit relation $\mathscr{L}_n[f](\cdot, t) \underset{n\to\infty}{\longrightarrow} \mathscr{L}[f](\cdot, t)$ in $L_p(\Omega)$ for any $t \in [0, T]$; therefore, $\mathscr{L}_n[f] \to \mathscr{L}[f]$ almost everywhere in Q_T. Consequently, if $f \in C^1(Q_T)$, then $\mathscr{L}[f]$ is measurable in Q_T.

Let us now prove the measurability of $\mathscr{L}[f]$ for an arbitrary function $f \in L_\vartheta(0, T; L_p(\Omega))$. Let $f_n \in C^1(Q_T)$, $n = 1, 2, \dots$, and let $f_n \to f$ in $L_\vartheta(0, T; L_p(\Omega))$. By virtue of the preceding considerations, the $\mathscr{L}[f_n]$, $n \geq 1$, are measurable in Q_T. By Definition 1.1, the linearity of Eq. (2.1), and assertion (i) of Lemma 2.1, the transform of the function $f_n - f$, $n \geq 1$, under the mapping $\mathscr{L}$ is the function $\mathscr{L}[f_n] - \mathscr{L}[f]$. Now the estimate of assertion (ii) of Lemma 2.1 implies the relation $\|\mathscr{L}[f_n](\cdot, t) - \mathscr{L}[f](\cdot, t)\|_{p,\Omega} = \|f_n(\cdot, t) - f(\cdot, t)\|_{p,\Omega}$ for almost all $t \in [0, T]$, which, together with an arbitrary choice of the sequence $\{f_n\}$, proves the convergence $\|\mathscr{L}[f_n](\cdot, t) - \mathscr{L}[f](\cdot, t)\|_{p,\Omega} \to 0$ as $n \to \infty$ for almost all $t \in [0, T]$. Consequently, $\mathscr{L}[f_n] \to \mathscr{L}[f]$ almost everywhere in Q_T. Thus, $\mathscr{L}[f]$ is a measurable function, which completes the proof of Proposition 1.1.

2.1.2. Additional Properties of the Lagrange Operator. Let us establish a number of properties of the Lagrange operator used in forthcoming considerations.

Proposition 2.1. (1) *If* $f \in L_\vartheta(0,T;L_1(\Omega))$, *then* $\int_\Omega \mathscr{L}[f]d\vec{x} = \int_\Omega f\,d\vec{x}$ *almost everywhere in* $[0,T]$;

(2) *if* $f_i \in L_{\vartheta_i}(0,T;L_{p_i}(\Omega))$, $i=1,\dots,k$, $\sum_{i=1}^k p_i^{-1} \le 1$, *and* $\sum_{i=1}^k \vartheta_i^{-1} \le 1$, *then*

$$\mathscr{L}[f_1\dots f_k] \in L_1(Q_T) \qquad \text{and} \qquad \mathscr{L}[f_1\dots f_k](\vec{x},t) = \mathscr{L}[f_1](\vec{x},t)\dots\mathscr{L}[f_k](\vec{x},t)$$

for almost all $(\vec{x},t) \in Q_T$;

(3) *if* $f_i \in L_1(Q_T)$, $i=1,\dots,k$, *then*

$$\mathscr{L}[f_1+\dots+f_k] \in L_1(Q_T) \qquad \text{and} \qquad \mathscr{L}[f_1+\dots+f_k](\vec{x},t) = \mathscr{L}[f_1](\vec{x},t)+\dots+\mathscr{L}[f_k](\vec{x},t)$$

for almost all $(\vec{x},t) \in Q_T$;

(4) *if* $f \in C^1(Q_T)$, $\vec{X}^0(\vec{x},t) = \vec{X}(\vec{x},t,0)$ *is a flow in the sense of Definition* 2.1, *and* $[f\circ\vec{X}^0](\vec{x},t) = f\left(\vec{X}^0(\vec{x},t),t\right)$, *then*

$$\mathscr{L}\left[f\circ\vec{X}^0\right](\vec{x},t) = f(\vec{x},t)$$

for almost all $(\vec{x},t) \in Q_T$;

(5) *if* $f \in C[0,T]$, *then* $\mathscr{L}[f](\vec{x},t) = f(t)$ *for any* $(\vec{x},t) \in Q_T$;

(6) *if* $\vec{v}_n, \vec{v} \in L_\gamma(0,T;W_\alpha^1(\Omega)\cap J(\Omega))$, $\vec{v}_n \to \vec{v}$ *in* $L_\gamma\left(0,T;W_{\alpha_1}^1(\Omega)\right)$, *where* $\alpha_1 < \infty$, $\alpha_1 \le \alpha$, $\mathscr{L}_n$ *and* $\mathscr{L}$ *are the Lagrange operators corresponding to the vector fields* $\vec{v}_n$ *and* $\vec{v}$, *respectively, and* $f \in L_\vartheta(0,T;L_p(\Omega))$, $1 < \vartheta, p$, *then* $\mathscr{L}_n[f] \to \mathscr{L}[f]$ $*$*-weakly in* $L_\vartheta(0,T;L_p(\Omega))$;

(7) *assertions* (1)–(3), (5), *and* (6) *of this proposition are also valid for* $\mathscr{L}^{-1}$.

The scheme of the proof of assertions (1)–(5) and (7) of Proposition 2.1 is the same as that of assertion (2) of Proposition 1.1; namely, we first justify the desired assertion for smooth $\vec{v}$ and f, and then its validity for nonsmooth $\vec{v}$ and f follows from assertion (iii) of Lemma 2.1 and Proposition 1.1. We present the proof of assertion (6) in detail. By the Banach–Steinhaus theorem [9, Chap. VII, Sec. 1, Th. 3], it suffices to show that first, $\sup_n \|\mathscr{L}_n[f]\|_{L_\vartheta(0,T;L_p(\Omega))} < \infty$, and second,

$$\int\limits_{Q_T} V(\vec{x},t)\mathscr{L}_n[f](\vec{x},t)\,d\vec{x}\,dt \to \int\limits_{Q_T} V(\vec{x},t)\mathscr{L}[f](\vec{x},t)\,d\vec{x}\,dt$$

for any V from some set dense in the space $L_{\vartheta'}(0,T;L_{p'}(\Omega))$, $p^{-1}+(p')^{-1}=1$, $\vartheta^{-1}+(\vartheta')^{-1}=1$.

The uniform boundedness of $\mathscr{L}_n$ in $L_\vartheta(0,T;L_p(\Omega))$ follows from the estimates of assertion (3) of Proposition 1.1. Now we construct a set of functions dense in $L_{\vartheta'}(0,T;L_{p'}(\Omega))$ and satisfying the last limit relation, which will complete the proof of assertion (6). By assertion (iii) of Lemma 1.1, we have $\|\mathscr{L}_n[f](t)-\mathscr{L}[f](t)\|_{p,\Omega} \to 0$ for almost all $t\in[0,T]$, and so $\mathscr{L}_n[f]\to\mathscr{L}[f]$ almost everywhere in Q_T. This, together with the Egorov theorem, means that, for each $\varepsilon>0$, there exists a set $\overline{Q}_T^\varepsilon \subset Q_T$ such that $\operatorname{meas} Q_T - \operatorname{meas} Q_T^\varepsilon < \varepsilon$ and $\mathscr{L}_n[f](\vec{x},t)\to\mathscr{L}[f](\vec{x},t)$ uniformly on $\overline{Q}_T^\varepsilon$, whence it follows that

$$\int\limits_E \mathscr{L}_n[f]d\vec{x}\,dt \to \int\limits_E \mathscr{L}[f]d\vec{x}\,dt,$$

where E is an arbitrary measurable subset of $\overline{Q}_T^\varepsilon$. Consider the numerical sequence $\varepsilon_k\to 0$, the corresponding sequence $Q_T^{\varepsilon_k}$ of domains, and the set of characteristic functions of all possible measurable subsets $E(\varepsilon_k)\subset Q_T^{\varepsilon_k}$, $k=1,2,\dots$ It remains to note that the linear span of this set is dense in $L_{\vartheta'}(0,T;L_{p'}(\Omega))$ [9, Chap. III, Sec. 3, Th. 4, Corollary 2].

3. THE LAGRANGIAN TRANSFORMATION OF EQUATION (1.1)

Let us clarify the meaning of generalized solutions of problems (1.1) and (1.3).

Definition 3.1. A function $u \in L_\delta(0,T;L_p(\Omega))$ is a *generalized solution* of the Cauchy problem (1.1) if the integral relation

$$\int_0^\tau dt \int_\Omega u\,(\partial_t\varphi + \vec{v}\cdot\nabla_x\varphi - c\varphi)\,d\vec{x} = \int_\Omega u\,(\vec{x},\tau)\,\varphi\,(\vec{x},\tau)\,d\vec{x} - \int_\Omega u_0\,(\vec{x})\,\varphi\,(\vec{x},0)\,d\vec{x} \tag{3.1}$$

is valid for all $\tau \in [0,T]$, where $\varphi\,(\vec{x},t)$ is a test function satisfying the condition $\varphi \in C^1(Q_T)$.

Definition 3.2. A function $U \in L_\delta(0,T;L_p(\Omega))$ is a *generalized solution* of the Cauchy problem (1.3) if the integral relation

$$\begin{aligned}\int_0^\tau dt \int_\Omega U\,(\vec{x},t)\,(\partial_t\Phi\,(\vec{x},t) - C\,(\vec{x},t)\,\Phi\,(\vec{x},t))\,d\vec{x}\\ = \int_\Omega U\,(\vec{x},\tau)\,\Phi\,(\vec{x},\tau)\,d\vec{x} - \int_\Omega U_0\,(\vec{x})\,\Phi\,(\vec{x},0)\,d\vec{x}\end{aligned} \tag{3.2}$$

is valid for all $\tau \in [0,T]$, where $\Phi(\vec{x},t)$ is a test function satisfying the condition $\Phi \in L_\infty(\Omega, C^1[0,T])$.

Note that the first integral on the right-hand side in (3.1) is well defined for any $\tau \in [0,T]$, since if u satisfies Eq. (1.1) in the sense of distributions, then $t \to u(t)$ is a weakly continuous mapping of the interval $[0,T]$ into the Lebesgue space $L_p(\Omega)$ [8, Chap. III, Sec. 1]. The same is true for relation (3.2).

3.1. Proof of Theorem 1.2

Let us first prove that if a function $u\,(\vec{x},t)$ is a generalized solution of the Cauchy problem for Eq. (1.1), then the function $U\,(\vec{x},t) = \mathscr{L}[u]\,(\vec{x},t)$ is a generalized solution of the Cauchy problem for Eq. (1.3). The proof is performed on the basis of a special choice of test functions in the integral relation (3.1). Let $\varphi_1 \in C_0^1(\Omega)$ and $\varphi_2 \in C^1[0,T]$. We set

$$\varphi_1^{\sigma\varepsilon}\,(\vec{x},t) = \left(\left(\varphi_1 \circ \left(\vec{X}^0 * \omega_\varepsilon\right)\right) * \bar{\omega}_\sigma\right)(\vec{x},t),$$

where $\vec{X}^0\,(\vec{x},t) = \vec{X}\,(\vec{x},t,0)$ is a flow (in the sense of Definition 2.1), $\omega_\varepsilon\,(\vec{x})$ is the averaging kernel occurring in assertion (iv) of Lemma 2.1, and $\bar{\omega}_\sigma(t) = \sigma^{-1}\bar{\omega}\,(t\sigma^{-1})$ is an averaging kernel. Here $\bar{\omega}$ is an even function from the class $D_+(\mathbf{R})$ with unit mean value. By Lemma 2.2, the set $\{\varphi_1^{\sigma\varepsilon}\,(\vec{x},t)\}_{\varepsilon,\sigma>0}$ is uniformly bounded in $L_\infty(Q_T)$, and

$$\begin{gathered}\varphi_1^{\sigma\varepsilon} \to \varphi_1 \circ \vec{X}^0 \quad \text{in} \quad L_\vartheta(Q_T), \qquad *\text{-weakly in} \quad L_\infty(Q_T),\\ \partial_t\varphi_1^{\sigma\varepsilon} + \vec{v}\cdot\nabla_x\varphi_1^{\sigma\varepsilon} \to 0 \quad \text{in} \quad L_\gamma(0,T;L_\beta(\Omega)),\\ \varphi_1^{\sigma\varepsilon}(\tau) \to \left(\varphi_1 \circ \vec{X}^0\right)(\tau), \qquad \varphi_1^{\sigma\varepsilon}(0) \to \left(\varphi_1 \circ \vec{X}^0\right)(0) \quad \text{in} \quad L_\vartheta(\Omega), \quad *\text{-weakly in} \quad L_\infty(\Omega)\end{gathered} \tag{3.3}$$

as $\varepsilon,\sigma \to 0$, where β is the exponent defined in assertion (iv) of Lemma 2.2, $\vartheta < \infty$ is an arbitrary exponent, and $\left(\varphi_1 \circ \vec{X}^0\right)(0) = \varphi_1\,(\vec{x})$ by the definition of a flow. We substitute $\Phi_{\sigma\varepsilon}\,(\vec{x},t) = \varphi_1^{\sigma\varepsilon}\,(\vec{x},t)\,\varphi_2(t)$ as a test function into (3.1) and pass to the limit; then, taking account of the assumptions of Theorem 1.2 and relation (3.3), we obtain

$$\begin{aligned}\int_0^\tau dt \int_\Omega u(\vec{x},t)\Big(\left(\varphi_1\circ\vec{X}^0\right)(\vec{x},t)\,\partial_t\varphi_2(t) - c(\vec{x},t)\left(\varphi_1\circ\vec{X}^0\right)(\vec{x},t)\,\varphi_2(t)\Big)\,d\vec{x}\\ = \int_\Omega u(\vec{x},\tau)\left(\varphi_1\circ\vec{X}^0\right)(\vec{x},\tau)\,\varphi_2(\tau)d\vec{x} - \int_\Omega u_0(\vec{x})\,\varphi_1(\vec{x})\,\varphi_2(0)d\vec{x}.\end{aligned} \tag{3.4}$$

Consider the transforms of the functions occurring in (3.4) under the mapping $\mathscr{L}$. We set

$$U(\vec{x},t) = \mathscr{L}[u](\vec{x},t), \qquad C(\vec{x},t) = \mathscr{L}[c](\vec{x},t). \tag{3.5}$$

By assertion (2) of Proposition 1.1 and assertion (4) of Proposition 2.1, we have

$$U\in L_\delta(0,T;L_p(\Omega)), \qquad C\in L_\gamma(0,T;L_q(\Omega)), \qquad \mathscr{L}\left[\varphi_1\circ\vec{X}^0\right](\vec{x},t) = \varphi_1(\vec{x}), \qquad \vec{x}\in\Omega. \tag{3.6}$$

Using formulas (3.5) and (3.6) and Proposition 2.1, we rewrite relation (3.4) in the equivalent form

$$\begin{aligned}\int_0^\tau dt \int_\Omega U(\vec{x},t)\left(\varphi_1(\vec{x})\,\partial_t\varphi_2(t) - C(\vec{x},t)\,\varphi_1(\vec{x})\,\varphi_2(t)\right)d\vec{x}\\ = \int_\Omega U(\vec{x},\tau)\,\varphi_1(\vec{x})\,\varphi_2(\tau)d\vec{x} - \int_\Omega U_0(\vec{x})\,\varphi_1(\vec{x})\,\varphi_2(0)d\vec{x}.\end{aligned} \tag{3.7}$$

[By the definition of the Lagrange transformation, $U_0(\vec{x}) = u_0(\vec{x})$ in (3.7).]

An arbitrary function $\Phi\in L_\infty(\Omega, C^1[0,T])$ can be approximated by functions Φ_n from the linear span of the set $\{\varphi_1(\vec{x})\,\varphi_2(t)\mid \varphi_1\in C_0^1(\Omega),\ \varphi_2\in C^1[0,T]\}$, so that $\Phi_n\to\Phi$ $*$-weakly in $L_\infty(\Omega, C^1[0,T])$. This, together with (3.6) and (3.7), implies (3.2). We have thereby shown that the function $U(\vec{x},t) = \mathscr{L}[u](\vec{x},t)$ is a generalized solution of the Cauchy problem for Eq. (1.3), which completes the proof of the first assertion of Theorem 2.1.

Let us prove the converse of Theorem 2.1, which claims that transforms of solutions of Eq. (1.3) under $\mathscr{L}^{-1}$ are solutions of Eq. (1.1). The proof is based on a special choice of test functions in the integral relation (3.2).

Let $\{\vec{v}_n\}\subset C^1([0,T], C_0^1(\Omega)\cap J(\Omega))$, $\vec{v}_n\to\vec{v}$ in $L_2\left(0,T;W^1_{\alpha_1}(\Omega)\right)$, where $\vec{v}$ is the vector field corresponding to the Lagrange operator $\mathscr{L}$, and let α_1 be an arbitrary number such that $\alpha_1<\infty$ and $\alpha_1\le\alpha$. By $\Phi_n^{0,t}$ we denote the operator of shift along trajectories of the vector field $\vec{v}_n$. Let $\varphi_1\in C^1[0,T]$ and $\varphi_2\in C_0^1(\Omega)$. Substituting the test function $\Phi(t,\vec{x}) = \varphi_1(t)\varphi_2\left(\Phi_n^{0,t}(\vec{x})\right)$ into the integral relation (3.2) and taking account of the fact that, by the definition of the operator of shift along trajectories given in Section 2.1,

$$d\Phi_n^{0,t}(\vec{x})/dt = \vec{v}_n\left(\Phi_n^{0,t}(\vec{x}), t\right), \tag{3.8}$$

we obtain the relation

$$\begin{aligned}\int_0^\tau dt \int_\Omega U(\vec{x},t)\Big(\partial_t\varphi_1(t)\varphi_2\left(\Phi_n^{0,t}(\vec{x})\right) + \varphi_1(t)\vec{v}_n\left(\Phi_n^{0,t}(\vec{x}),t\right)\cdot\left(\nabla\varphi_2\circ\Phi_n^{0,t}\right)(\vec{x})\\ - C(\vec{x},t)\,\varphi_1(t)\varphi_2\left(\Phi_n^{0,t}(\vec{x})\right)\Big)d\vec{x}\\ = \int_\Omega U(\vec{x},\tau)\,\varphi_1(\tau)\varphi_2\left(\Phi_n^{0,\tau}(\vec{x})\right)d\vec{x} - \int_\Omega U_0(\vec{x})\,\varphi_1(0)\varphi_2(\vec{x})\,d\vec{x}.\end{aligned} \tag{3.9}$$

By virtue of assertion (4) of Proposition 1.1, we have

$$\varphi_2\left(\Phi_n^{0,t}(\vec{x})\right) = \mathscr{L}_n[\varphi_2](\vec{x},t), \qquad \left(\nabla\varphi_2 \circ \Phi_n^{0,t}\right)(\vec{x}) = \mathscr{L}_n[\nabla\varphi_2](\vec{x},t),$$

and $\vec{v}_n\left(\Phi_n^{0,t}(\vec{x}),t\right) = \mathscr{L}_n[\vec{v}_n](\vec{x},t)$. We substitute these expressions into (3.9) and pass to the limit as $n \to \infty$. By assertion (3) of Proposition 1.1, we obtain

$$\begin{aligned}&\mathscr{L}_n[\vec{v}_n](\vec{x},t)\,\mathscr{L}_n[\nabla\varphi_2](\vec{x},t)\\ &\quad = \mathscr{L}_n[\vec{v}_n - \vec{v}](\vec{x},t)\,\mathscr{L}_n[\nabla\varphi_2](\vec{x},t) + \mathscr{L}_n[\vec{v}](\vec{x},t)\,\mathscr{L}_n[\nabla\varphi_2](\vec{x},t) \quad \text{almost everywhere in} \quad Q_T.\end{aligned}$$

Since, by assertion (3) of Proposition 1.1,

$$\left\|\mathscr{L}_n[\partial_j v_{ni} - \partial_j v_i]\right\|_{L_\gamma(0,T;L_{\alpha_1}(\Omega))} = \left\|\partial_j v_{ni} - \partial_j v_i\right\|_{L_\gamma(0,T;L_{\alpha_1}(\Omega))}, \qquad i,j = 1,2,$$

we have $\mathscr{L}_n[\vec{v}_n - \vec{v}] \to 0$ in $L_\gamma\left(0,T;W^1_{\alpha_1}(\Omega)\right)$. It follows from assertion (6) of Proposition 2.1 that $\mathscr{L}_n[\varphi_2] \to \mathscr{L}[\varphi_2]$ $*$-weakly in $L_\infty(Q_T)$.

Finally, from assertions (2) and (7) of Proposition 2.1, we have $\mathscr{L}_n[\vec{v}]\,\mathscr{L}_n[\nabla\varphi_2] \to \mathscr{L}[\vec{v}\cdot\nabla\varphi_2]$ either weakly in $L_\gamma(0,T;L_{\alpha_2}(\Omega))$ if $\alpha < N$, where the exponent α_2 is found from the condition $W^1_\alpha(\Omega) \subset L_{\alpha_2}(\Omega)$ (continuously) by the Sobolev embedding theorem, i.e., $\alpha_2 \geq \alpha N(N-\alpha)^{-1}$, or $*$-weakly in $L_\gamma(0,T;L_\infty(\Omega))$ if $\alpha \geq N$, since $W^1_\alpha(\Omega) \subset L_\infty(\Omega)$ (continuously) in this case.

Taking account of the preceding considerations and setting $u(\vec{x},t) = \mathscr{L}^{-1}[U](\vec{x},t)$, from (3.9), we obtain

$$\begin{aligned}&\int_0^\tau dt \int_\Omega \mathscr{L}\left[u\left(\varphi_2\partial_t\varphi_1 + (\vec{v}\cdot\nabla_x\varphi_2)\,\varphi_1 - c\varphi_1\varphi_2\right)\right](\vec{x},t)\,d\vec{x}\\ &\qquad = \int_\Omega \mathscr{L}[u\varphi_1\varphi_2](\vec{x},\tau)\,d\vec{x} - \int_\Omega u_0(\vec{x})\,\varphi_1(0)\varphi_2(\vec{x})\,d\vec{x}.\end{aligned} \tag{3.10}$$

By virtue of assertion (1) of Proposition 2.1, the last relation is equivalent to relation (3.1) with a test function of the form $\varphi(\vec{x},t) = \varphi_1(t)\varphi_2(\vec{x})$. To complete the proof of Theorem 1.2, it remains to note that the linear span of the set of such functions is dense in $C^1_0(\Omega) \times C^1[0,T]$.

4. PROOF OF THEOREM 1.1

In this section, we prove that problem (1.3) has at most one generalized solution, which, together with Theorem 1.2, will imply the assertion of Theorem 1.1. The proof is performed on the basis of the following auxiliary assertion.

Proposition 4.1. *Let $U(\vec{x},t)$ be a generalized solution of the Cauchy problem for Eq.* (1.3) *in the sense of Definition* 3.2. *Then the relation*

$$\int_0^\tau U(\partial_t h - Ch)\,dt = U(\vec{x},\tau)\,h(\vec{x},\tau) - U_0(\vec{x})\,h(\vec{x},0) \tag{4.1}$$

is valid for any $\tau \in [0,T]$, for almost all $\vec{x} \in \Omega$, and for an arbitrary test function $h(\vec{x},t)$ such that $h(\vec{x},t)$ is jointly measurable with respect to the variables $(\vec{x},t)$ in Q_T and $h(\vec{x},\cdot) \in W^1_\gamma(0,T)\cap C[0,T]$ for almost all $\vec{x} \in \Omega$.

Proof. In the integral relation (3.2), we set $\Phi(\vec{x},t) = g(\vec{x})\,f(\vec{x},t)$, where $g \in L_\infty(\Omega)$ and $f \in L_\infty(\Omega, C^1[0,T])$. Since g is arbitrary, we find that

$$\int_0^\tau U(\partial_t f - Cf)\,dt = U(\vec{x},\tau)\,f(\vec{x},\tau) - U_0(\vec{x})\,f(\vec{x},0) \tag{4.2}$$

for any $\tau \in [0,T]$ and for almost all $\vec{x} \in \Omega$.

It remains to show that, for any function $h(\vec{x},t)$ satisfying the assumptions of Proposition 4.1, there exists a sequence $\{f^k\} \subset L_\infty(\Omega, C^1[0,T])$ such that

$$\begin{gathered} f^k \to h \quad \text{almost everywhere in} \quad Q_T, \\ f^k(\vec{x},\cdot) \to h(\vec{x},\cdot) \quad \text{in} \quad C[0,T]\cap W^1_\gamma(0,T) \quad \text{for almost all} \quad \vec{x}\in\Omega. \end{gathered} \tag{4.3}$$

Consider the system $\{\varphi_i\}$ of trigonometric functions, which is complete and orthogonal in $L_2(0,T)$. Let $S_k(\vec{x},t) = c_1(\vec{x})\varphi_1(t) + \cdots + c_k(\vec{x})\varphi_k(t)$, $k = 1,2,\ldots$, be the sequence of partial sums of the Fourier series of the function $h(\vec{x},t)$, where $c_i(\vec{x}) = (2/T)\int_0^T h(\vec{x},\vartheta)\varphi_i(\vartheta)d\vartheta$, $i = 1,2,\ldots,k$, are the Fourier coefficients, and let $C_k(\vec{x},t) = (S_0(\vec{x},t) + \cdots + S_k(\vec{x},t))(k+1)^{-1}$, $k = 1,2,\ldots,k$, be the sequence of Cesaro sums of the function $h(\vec{x},t)$. Obviously, $C_k(\vec{x},\cdot) \in C^1[0,T]$ for almost all $\vec{x}\in\Omega$, $k \ge 1$; the product $h(\vec{x},\cdot)\varphi_i(\cdot)$ is integrable on $(0,T)$ for almost all $\vec{x}\in\Omega$ [10, Chap. I, item 4]; consequently, the $c_i(\vec{x})$ are measurable in Ω, and so the C_k are measurable in Q_T.

By the Luzin theorem, for any positive integer n, there exists a closed set $\bar{\Omega}^i_n$,

$$\operatorname{meas}\bar{\Omega}^i_n > \operatorname{meas}\Omega - n^{-1},$$

such that the function $c_i(\vec{x})$ is continuous on $\bar{\Omega}^i_n$, $i = 1,2,\ldots$ We set $S^k_l = \sum_{i=0}^l c_i^{k^3}(\vec{x})\varphi_i(t)$ and $f^k = \left(S^k_0 + \cdots + S^k_k\right)(k+1)^{-1}$, where $c^n_i(\vec{x}) = c_i(\vec{x})$ for $\vec{x}\in\bar{\Omega}^i_n$ and $c^n_i(\vec{x}) = 0$ for $\vec{x}\in\Omega\backslash\bar{\Omega}^i_n$, $i,j = 0,1,\ldots$, $n = 1,2,\ldots$ Note that $f^k \in L_\infty(\Omega; C^1[0,T])$, $k = 1,2,\ldots$, and $f^k(\vec{x},t) = C_k(\vec{x},t)$ for $\vec{x}\in\bigcap_{i=1}^k \bar{\Omega}^i_{k^3}$.

We set $\bar{\Omega}^n = \bigcap_{k=n}^\infty \left(\bigcap_{i=1}^k \bar{\Omega}^i_{k^3}\right)$, $n\in\mathbf{N}$. We have

$$\operatorname{meas}\bigcap_{i=1}^k \bar{\Omega}^i_{k^3} \ge \operatorname{meas}\Omega - \sum_{i=1}^k \operatorname{meas}\left(\Omega\backslash\bar{\Omega}^i_{k^3}\right) = \operatorname{meas}\Omega - k^{-2};$$

$$\operatorname{meas}\bigcap_{k=n}^\infty\left(\bigcap_{i=1}^k \bar{\Omega}^i_{k^3}\right) \ge \operatorname{meas}\Omega - \sum_{k=n}^\infty k^{-2} = \operatorname{meas}\Omega - n^{-1}; \qquad f^k(\vec{x},t) = C_k(\vec{x},y,t)$$

for $\vec{x}\in\bar{\Omega}_n$ and for any $k \ge n$. Since $C_k(\vec{x},\cdot) \to h(\vec{x},\cdot)$ in $C[0,T]\cap W^1_\gamma(0,T)$ for almost all $\vec{x}\in\Omega$, it follows from the last relation, the Fejer theorem, and its corollary for integrable functions [11, Chap. 5, item 3.1; Chap. 6, item 1.1] that

$$\begin{gathered} f^k \to h \quad \text{almost everywhere in} \quad \bar{\Omega}_n\times[0,T], \\ f^k(\vec{x},\cdot) \to h(\vec{x},\cdot) \quad \text{in} \quad C[0,T]\cap W^1_\gamma(0,T) \quad \text{for any} \quad \vec{x}\in\bar{\Omega}_n. \end{gathered}$$

Therefore, by virtue of an arbitrary choice of $n\in\mathbf{N}$ and the estimate for the measure of the set $\bar{\Omega}_n$, relation (4.3) is valid for the sequence $\{f^k\}_{k=1}^\infty$.

Let us proceed to the proof of Theorem 1.1.

Let $U(\vec{x},t)$ be a generalized solution of the Cauchy problem for Eq. (1.3). Since Eq. (1.3) is linear, it follows that the uniqueness of the solution $U(\vec{x},t)$ is equivalent to the fact that if $U_0(\vec{x}) = 0$ for almost all $\vec{x}\in\Omega$, then $U(\vec{x},t) = 0$ for almost all $(\vec{x},t)\in Q_T$. To prove this fact, we note that, by Proposition 4.1, $U(\vec{x},t)$ satisfies relation (4.1), which, in view of the assumption $U_0(\vec{x}) = 0$, acquires the form

$$\int_0^\tau U(\partial_t h - Ch)\,dt = U(\vec{x},\tau)\,h(\vec{x},\tau). \tag{4.4}$$

Consider the function $\bar{h}(\vec{x},t) = \exp\left(\int_\tau^t C(\vec{x},s)\,ds\right)$. Since $C(\vec{x},\cdot)\in L_\gamma(0,T)$ for almost all $\vec{x}\in\Omega$ and $C(\vec{x},t)$ is jointly measurable, it follows that $\bar{h}(\vec{x},t)$ is an admissible test function for the integral relation (4.1) and its corollary (4.4). Substituting $\bar{h}(\vec{x}, t)$ into the integral relation (4.4) and taking into account an arbitrary choice of $\tau \in [0, T]$, we obtain $U(\vec{x}, t) = 0$ almost everywhere in Q_T, which completes the proof of Theorem 1.1.

Remark 4.1. The assumptions of Theorem 1.1 imposed on the coefficient $c(\vec{x}, t)$ and providing the uniqueness of the solution of the Cauchy problem for Eq. (1.1) are less restrictive than those in [1–3], which provided its solvability for an arbitrary choice of Cauchy data from $L_p(\Omega)$. We thereby naturally encounter the problem as to whether there exists an equation of the form (1.1) that satisfies the assumptions of Theorem 1.1, does not satisfy the conditions in [1–3], and admits a solution in Lebesgue classes for some choice of the Cauchy data $u_0(\vec{x}) \not\equiv 0$ (the case $u_0 \equiv 0$ is obvious). The following example answers this question.

Consider the equation

$$u_t + v_1 \partial_1 u + v_2 \partial_2 u + cu = 0 \tag{4.5}$$

with coefficients $c = -1 - v_1 - v_2$ and $\{v_1, v_2\} = \mathrm{rot}_x H$, where $H \in \overset{\circ}{W}{}_2^2(B^2)$ (in mechanics, H is referred to as the flux function), $\mathrm{rot}_x = \{\partial_2, -\partial_1\}$, and $B^2 = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$ is the unit disk in $\mathbf{R}^2$. Obviously, $\partial_1 v_1 + \partial_2 v_2 = 0$ almost everywhere in B^2, $v_1, v_2 \in \overset{\circ}{W}{}_2^1(B^2)$, $c \notin L_\infty(B^2)$, and $c \in L_q(B^2)$, where $q < \infty$ is an arbitrary exponent. Therefore, the assumptions of Theorem 1.1 are valid, the conditions in [1–3] fail, but the Cauchy problem for Eq. (4.5) in the space-time cylinder $Q = B^2 \times [0, 1]$ with the Cauchy data $u|_{t=0} = e^{x_1 + x_2} \in L_\infty(B^2)$ has the solution $u(t, x_1, x_2) = e^{t + x_1 + x_2} \in L_\infty(Q)$.

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