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MATHEMATICS

Cauchy Problem for the Graetz–Nusselt Ultraparabolic Equation

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We prove the unique solvability of an entropy formulation of the Cauchy problem for a quasilinear ultraparabolic anisotropic diffusion equation with nonsmooth convection coefficients. For this purpose, a well-posed kinetic formulation of the problem is constructed, which arises from examining the properties of Young measures associated with a sequence of solutions to approximate uniformly parabolic problems. The central point in our study is a renormalization of the resulting kinetic equation, which yields a renormalized inequality. The main results of this paper are derived by analyzing the structure of this inequality.

In a spatiotemporal layer $\mathbb{R}^d \times (0, T)$, we consider the following Cauchy problem for a diffusion–convection equation with initial data belonging to $L^\infty(\mathbb{R}^d)$ and with the one-periodicity condition with respect to spatial variables:

$$\partial_t u + \mathrm{div}_x(\mathbf{v}a(u)) - \mathrm{div}_x(A\nabla_x b(u)) = 0, \tag{1}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \text{ a.e. in } \mathbf{x} \in \mathbb{R}^d, \tag{2}$$

$$u(\mathbf{x} + \mathbf{e}_i, t) = u(\mathbf{x}, t) \quad \text{a.e. in } (\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]. \tag{3}$$

It is assumed that $u_0 \in L^\infty(\mathbb{R}^d)$ is a given function satisfying

$$0 \le u_0(\mathbf{x}) \le 1, \quad u_0(\mathbf{x} + \mathbf{e}_i) = u_0(\mathbf{x}, t) \text{ a.e. in } \mathbf{x} \in \mathbb{R}^d. \tag{4}$$

Here, $\mathbf{e}_i (i = 1, 2, \ldots, d)$ are the vectors of the Cartesian basis in $\mathbb{R}^d$, $u(\mathbf{x}, t)$ is the function sought, and $A \ge 0$ is a given symmetric positive semidefinite matrix. It is also assumed that a, b, and $\mathbf{v}$ satisfy the conditions

$$a \in C^1_{\mathrm{loc}}(\mathbb{R}), \quad b \in C^2_{\mathrm{loc}}(\mathbb{R}); \quad b'(u) > 0 \text{ for all } u \in \mathbb{R}; \tag{5}$$

$$\mathbf{v}, \nabla_x \mathbf{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times [0, T]); \ \mathbf{v}(\mathbf{x} + \mathbf{e}_i, t) = \mathbf{v}(\mathbf{x}, t), \quad \mathrm{div}_x \mathbf{v}(\mathbf{x}, t) = 0 \text{ a.e. in } \mathbb{R}^d \times [0, T]. \tag{6}$$

The matrix A maps $\mathbb{R}^d$ into the space $\mathcal{L} := \Im(A) \subset \mathbb{R}^d$ of dimension $k := \mathrm{rank} A$. If $k < d$, then Eq. (1) is ultraparabolic. Such equations arise in fluid dynamics, combustion theory, and financial mathematics (see [1]). In particular, they describe unsteady mass or heat transport when the effect of diffusion in some spatial directions is negligible as compared with the effect of convection [2]. Such equations were considered for the first time by Graetz [3] and Nusselt [4].

In this paper, we use the kinetic equation method to prove the existence and uniqueness of an entropy solution to problem (1)–(3). To introduce the concept of an entropy solution, we need some notation. In what follows, Ω denotes the unit cube $(0, 1)^d$; Q is the cylinder $\Omega \times (0, T)$; $L^p \subset L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and $H^{s,p} \subset H^{s,p}_{\mathrm{loc}}(\mathbb{R}^d)$ are Banach spaces consisting of one-periodic functions and equipped with the norms $\|u\|_{L^p} = \|u\|_{L^p(\Omega)}$ and $\|u\|_{H^{s,p}} = \|u\|_{H^{s,p}(\Omega)}$, respectively; and C^l $(l \ge 0)$ is a closed subspace consisting of functions $u \in C^l(\mathbb{R}^d)$ that are one-periodic with respect to x_i, where $1 \le i \le d$.

The differential operator $\mathbf{A} = \mathrm{div}_x(A\nabla_x \cdot): C^\infty \mapsto L^2$ is symmetric and nonnegative in the Hilbert space L^2 and has a self-adjoint extension $\mathbf{A}: D(\mathbf{A}) \mapsto L^2$. To describe the domain $D(\mathbf{A})$, we note that $A = O^*\mathcal{D}O$, $\mathcal{D} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0\}$, and $O^*O = \mathbf{I}$, where λ_i are positive numbers. For an arbitrary function $u \in L^2$, a function $w \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and a vector field $\partial w \in H^{-1,2}_{\mathrm{loc}}(\mathbb{R}^d)$ are defined by the formulas $w(\mathbf{x}) = u(O\mathbf{x})$ and $\partial w = \{\partial_{x_1} w, \partial_{x_2} w, \ldots, \partial_{x_k} w, 0, \ldots, 0\}^T$. A function $u \in L^2$ belongs to $D(\mathbf{A})$ if and only if $w \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $\partial w \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Endowed with the norm

$$\|u\|^2_{\mathfrak{H}} = \|u\|^2_{L^2} + \left\|A^{\frac{1}{2}}\nabla_x u\right\|^2_{L^2},$$

$D(\mathbf{A})$ becomes a Hilbert space, which is hereafter denoted by $\mathfrak{H}$. Now, we can define the concept of an entropy solution to problem (1)–(3).

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Definition 1. A function $u \in L^\infty \cap L^2(0, T; \mathfrak{H})$ is an entropy solution to problem (1)–(3) if and only if

$$\int_Q \left\{ \varphi(u)\partial_t\zeta + \psi(u)\mathbf{v}\cdot\nabla_x\zeta + \omega(u)\mathrm{div}_x(A\nabla_x\zeta) - \varphi''(u)b'(u)\left|A^{\frac{1}{2}}\nabla_x u\right|^2\zeta \right\} d\mathbf{x}\,dt + \int_\Omega \varphi(u_0)\zeta(\mathbf{x}, 0)d\mathbf{x} \geq 0$$

for any functions φ, ψ, and ω such that $\varphi \in C^2_{\mathrm{loc}}(\mathbb{R})$, $\varphi''(u) \geq 0$, $\psi'(u) = a'(u)\varphi'(u)$, and $\omega'(u) = b'(u)\varphi'(u)$ and for any nonnegative functions $\zeta \in C^2_{\mathrm{loc}}(\mathbb{R}^d \times [0, T])$ that are one-periodic with respect to **x** and vanish at $t = T$.

Below is the main result of this paper.

Theorem 1. *For an arbitrary initial function $u_0 \in L^\infty$, problem* (1)–(3) *has a unique entropy solution.*

This theorem is proved by applying the kinetic equation method, which reduces a quasilinear equation to a linear one for a "distribution function" that involves additional kinetic variables (see, e.g., [5–7]). In this work, we propose a kinetic formulation of problem (1)–(3) that makes it possible to simultaneously examine measure-valued and entropy solutions.

KINETIC FORMULATION

In addition to the original problem in $\mathbb{R}^d \times (0, T)$, we consider its regularization

$$\partial_t u_\varepsilon + \mathrm{div}_x(\mathbf{v}_\varepsilon a_\varepsilon(u_\varepsilon)) - \mathrm{div}_x(A\nabla_x b(u_\varepsilon)) = \varepsilon\Delta_x u_\varepsilon \quad (7)$$

when supplemented with conditions (2) and (3). Here, $\mathbf{v}_\varepsilon \in C^\infty(0, T; C^\infty)$ is a solenoidal vector field and $a_\varepsilon \in C^\infty(\mathbb{R})$ $(\varepsilon > 0)$ is a smooth function that satisfy the respective limit relations $\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{L^1(0, T; H^{1,1})} \to 0$ and $\|a_\varepsilon - a\|_{H^{1,1}(0, 1)} \to 0$ as $\varepsilon \searrow 0$.

The theory of second-order parabolic equations [8] implies that the problem given by (7), (2), and (3) has a unique smooth solution satisfying the inequalities

$$0 \leq u_\varepsilon \leq 1, \quad \|u_\varepsilon\|_{L^2(0, T; \mathfrak{H})} \leq c_0, \quad (8)$$

where c_0 is a constant independent of ε.

Let $\mathbb{M}(\mathbb{R}^n)$ denote the Banach space of bounded Radon measures on $\mathbb{R}^n$. A mapping ν: $\mathbb{R}^d_x \times (0, T) \mapsto \mathbb{M}(\mathbb{R}^n)$ is called bounded, weakly measurable, and one-periodic with respect to **x** if, for any function $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d_x \times (0, T); C_0(\mathbb{R}^n))$, the mapping $(\mathbf{x}, t) \mapsto \int_{\mathbb{R}^n_p} F(\mathbf{x}, t, p)\,d\nu_{x, t}(p)$ is measurable and

$$\int_{\mathbb{R}^n_p} F(\mathbf{x}, t, p)d\nu_{x+e_i, t}(p) = \int_{\mathbb{R}^n_p} F(\mathbf{x} - \mathbf{e}_i, t, p)d\nu_{x, t}(p) \quad \text{for } i = 1, 2, \ldots, d.$$

According to [9], the parametrized measure ν belongs to $L^\infty_w(\mathbb{R}^d_x \times (0, T); \mathbb{M}(\mathbb{R}^n))$.

It follows from (8), Tartar's theorem [10], and Ball's theorem [11] that there exists a subsequence $\{u_\varepsilon\}$ and measure-valued functions $\mu \in L^\infty_w(\mathbb{R}^d_x \times [0, T]; \mathbb{M}(\mathbb{R}_\lambda))$ and $\sigma \in L^\infty_w(\mathbb{R}^d_x \times [0, T]; \mathbb{M}(\mathbb{R}_\lambda \times \mathscr{L}_q))$ that are one-periodic in **x** and such that, for any function $g \in C(\mathbb{R}_\lambda)$, it is true that

$$g(u_\varepsilon) \xrightarrow[\varepsilon \searrow 0]{} g^* \text{ weakly star in } L^\infty(Q), \quad \text{where } g^* = \int_{\mathbb{R}_\lambda} g(\lambda)d\mu_{x, t}(\lambda), \quad (9)$$

and, for any function $h \in C(\mathbb{R}_\lambda \times \mathscr{L}_q)$ satisfying $|h(\lambda, \mathbf{q})| \leq c(1 + |\lambda| + |\mathbf{q}|)^k$, $0 \leq k < 2$, it is true that

$$h\left(u_\varepsilon, A^{\frac{1}{2}}\nabla_x u_\varepsilon\right) \xrightarrow[\varepsilon \searrow 0]{} h^* \text{ weakly in } L^r(Q), \quad \text{where } h^* = \int_{\mathbb{R}_\lambda \times \mathscr{L}_q} h(\lambda, \mathbf{q})d\sigma_{x, t}(\lambda, \mathbf{q}) \quad (10)$$

for $1 < r \leq \frac{2}{k}$.

The measures $\mu_{x, t}$ and $\sigma_{x, t}$ are called Young measures and are associated with the weakly convergent sequences $\{u_\varepsilon\}$ and $\{u_\varepsilon, A^{1/2}\nabla_x u_\varepsilon\}$, respectively. Let f denote the distribution function of $\mu_{x, t}$:

$$f(\mathbf{x}, t, \lambda) = \int_{\mathbb{R}_s} 1_{\lambda \geq s}d\mu_{x, t}(s).$$

Problem K [kinetic formulation of problem (1)–(3)]. *Let* f_0: $\mathbb{R}^d_x \times \mathbb{R}_\lambda \mapsto [0, 1]$ *be a measurable function that is one-periodic with respect to x, monotonic and right continuous with respect to* λ, *and satisfies the condition*

$$f_0(\mathbf{x}, \lambda) = \begin{cases} 0 & \text{if } \lambda < 0 \\ 1 & \text{if } \lambda \geq 1 \end{cases} \quad \text{a.e. in } \mathbf{x} \in \mathbb{R}^d. \quad (11)$$

It is necessary to find a distribution function $f \in L^\infty(\mathbb{R}^d_x \times (0, T) \times \mathbb{R}_\lambda)$, a parametrized nonnegative measure $\sigma \in L^\infty_w(\mathbb{R}^d_x \times (0, T); \mathbb{M}(\mathbb{R}_\lambda \times \mathcal{L}_q))$, and a nonnegative defect measure $M \in \mathbb{M}(\mathbb{R}^d_x \times (0, T) \times \mathbb{R}_\lambda)$ that satisfy the following conditions:

(a) $f(\mathbf{x}, t, \lambda)$ is a one-periodic function of $\mathbf{x}$ and a monotonic and right continuous function of $\lambda \in \mathbb{R}$. Moreover,

$$f(\mathbf{x}, t, \lambda) = \begin{cases} 0 & \text{if } \lambda < 0 \\ 1 & \text{if } \lambda \geq 1. \end{cases}$$

In particular, $0 \leq f \leq 1$ a.e. in $Q \times \mathbb{R}_\lambda$. This means that the Stieltjes measure $\mu_{x,t} = d_\lambda f(\mathbf{x}, t, \lambda)$ is a probability measure on $\mathbb{R}_\lambda$ and $\operatorname{spt}\mu_{x,t} \subset [0, 1]$.

(b) The support of the parametrized measure $\sigma_{x,t}$ lies in the strip $[0, 1]_\lambda \times \mathcal{L}_q$. The mapping $(\mathbf{x}, t) \mapsto \sigma_{x,t}$ is one-periodic with respect to $\mathbf{x}$ and satisfies

$$\int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} d\sigma_{x,t}(\lambda, \mathbf{q}) = 1,$$

$$\int\limits_Q \left\{ \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} |\mathbf{q}|^2 \, d\sigma_{x,t}(\lambda, \mathbf{q}) \right\} d\mathbf{x} \, dt < \infty.$$

In particular, the function χ, defined as

$$\chi(\mathbf{x}, t, s) = \int\limits_{(-\infty, s] \times \mathcal{L}_q} |\mathbf{q}|^2 \, d\sigma_{x,t}(\lambda, \mathbf{q}),$$

is a one-periodic function of $\mathbf{x}$ and a monotonic and right continuous function of s, and the support of the Stieltjes measure $d_\lambda\chi(\mathbf{x}, t, \lambda)$ lies on $[0, 1]$ a.e. in $(\mathbf{x}, t) \in \mathbb{R}^d_x \times (0, T)$.

(c) For any $g \in C^1_{\text{loc}}(\mathbb{R}_\lambda)$, the function

$$G\colon (\mathbf{x}, t) \mapsto \int\limits_{\mathbb{R}_\lambda} g(\lambda) d_\lambda f(\mathbf{x}, t, \lambda)$$

belongs to the Hilbert space $L^2(0, T; \mathfrak{H})$, and

$$A^{\frac{1}{2}} \nabla_x G(\mathbf{x}, t) = \int\limits_{\mathbb{R}_\lambda \times \mathcal{L}_q} g'(\lambda) \mathbf{q} \, d\sigma_{x,t}(\lambda, q)$$

holds a.e. in $(\mathbf{x}, t) \in \mathbb{R}^d_x \times (0, T)$.

(d) The defect measure M is one-periodic with respect to $\mathbf{x}$.

(e) The distribution function f satisfies the following equation and initial condition:

$$\partial_t f + \operatorname{div}_x(a'(\lambda) f \mathbf{v} - b'(\lambda) A \nabla_x f) + \partial_\lambda(b'(\lambda)\partial_\lambda \chi + M) = 0, \tag{12}$$

$$f(\mathbf{x}, 0, \lambda) = f_0(\mathbf{x}, \lambda). \tag{13}$$

Formulas (12) and (13) are understood in the sense of distributions and can be equivalently represented as the integral equality

$$\int\limits_{Q \times \mathbb{R}_\lambda} \{\partial_t \zeta + a'(\lambda) \mathbf{v} \cdot \nabla_x \zeta + b'(\lambda) \operatorname{div}_x(A \nabla_x \zeta)\} \times f(\mathbf{x}, t, \lambda) d\mathbf{x} \, dt \, d\lambda + \int\limits_{Q \times \mathbb{R}_\lambda} \partial_\lambda \zeta \, dM + \int\limits_{Q \times \mathbb{R}_\lambda} b'(\lambda) \partial_\lambda \zeta \, d_\lambda \chi \, d\mathbf{x} \, dt + \int\limits_{\Omega \times \mathbb{R}_\lambda} \zeta(\mathbf{x}, 0, \lambda) f_0(\mathbf{x}, \lambda) d\mathbf{x} \, d\lambda = 0,$$

in which $\zeta(\mathbf{x}, t, \lambda)$ is an arbitrary one-periodic smooth function of $\mathbf{x}$ that vanishes in the neighborhood of the plane $\{t = T\}$ for sufficiently large $|\lambda|$.

It is easy to see that the solution set of the problem K is convex. The general theory of Young measures [9], Eq. (7), conditions (2) and (3), and limit relations (9) and (10) imply the following result.

Theorem 2. *Suppose that* $f_0\colon \mathbb{R}^d_x \times \mathbb{R}_\lambda \mapsto [0, 1]$ *is an arbitrary function that is one-periodic in* $\mathbf{x}$, *monotonic and right continuous in* λ, *and satisfies the equality*

$$f_0(\mathbf{x}, \lambda)(1 - f_0(\mathbf{x}, \lambda)) = 0 \tag{14}$$

and condition (11). *Then, the problem* K *has at least one solution* (f, σ, M) *with the initial data* f_0.

The relationship between the entropy solutions to problem (1)–(3) and the solutions to the problem K is established by the following theorem.

Theorem 3. *If u is an entropy solution to problem* (1)–(3) *with the initial data* u_0, *then the problem* K *with the initial data*

$$f_0(\mathbf{x}, \lambda) = \begin{cases} 0 & \text{if} \quad \lambda < u_0(\mathbf{x}) \\ 1 & \text{if} \quad \lambda \geq u_0(\mathbf{x}) \end{cases} \tag{15}$$

has a solution such that

$$f(\mathbf{x}, t, \lambda) = \begin{cases} 0 & \text{if} \quad \lambda < u(\mathbf{x}, t) \\ 1 & \text{if} \quad \lambda \geq u(\mathbf{x}, t). \end{cases}$$

Conversely, if (f, σ, M) *is a solution to the problem* K *with initial data* (15) *and f takes only the values* 0 *and* 1, *then* $u(\mathbf{x}, t) = \sup\{\lambda\colon f(\mathbf{x}, t, \lambda) = 0\}$ *is an entropy solution to problem* (1)–(3) *with the initial data* $u_0(\mathbf{x})$.

The solution to the problem K can be called a measure-valued solution to problem (1)–(3) in view of the well-known concept of measure-valued solutions to scalar conservation laws [9, Chapter 4; 10].

RENORMALIZATION

Since Eq. (12) is linear, we can use the idea of renormalization suggested in [12] and justify a renormalization procedure for (12), thus yielding the following result.

Theorem 4. *For any smooth convex function* φ *on* [0, 1], *there exists a Borel measure* $H_\varphi \in C(\mathbb{R}_\lambda \times Q)^*$ *with a support in the strip* $\{0 \le \lambda \le 1\}$ *such that*

$$\int\limits_{\mathbb{R}_\lambda \times Q} \varphi(f)\{\partial_t \zeta + a'(\lambda)\mathbf{v} \cdot \nabla_x \zeta + b'(\lambda)\operatorname{div}_x(A\nabla_x \zeta)\}$$

$$\times d\mathbf{x}\, dt\, d\lambda + \int\limits_{\mathbb{R}_\lambda \times \Omega} \varphi(f_0)\zeta(\mathbf{x}, 0, \lambda) d\mathbf{x}\, d\lambda$$

$$- \int\limits_{\mathbb{R}_\lambda \times Q} \partial_\lambda \zeta\, dH_\varphi \le 0 \qquad (16)$$

for any nonnegative smooth function $\zeta(\mathbf{x}, t, \lambda)$ *that is one-periodic with respect to* $\mathbf{x}$ *and vanishes in the neighborhood of the plane* $\{t = T\}$ *for sufficiently large* $|\lambda|$.

To conclude, we explain how renormalized inequality (16) can be used to prove Theorem 1.

Setting $\varphi(f) = f(f - 1)$ and $\zeta(\mathbf{x}, t, \lambda) = \zeta_1(\lambda)\zeta_2(t)$ in (16), where ζ_1 is a nonnegative function equal to unity on [0, 1] and ζ_2 is a nonnegative function that vanishes at $t = T$ and monotonically decreases for $t < T$, we conclude that $f(f - 1) \equiv 0$. The existence statement in Theorem 1 follows from this result, item (a) in the statement of the problem K, and Theorem 3. Now, we assume that the problem K has two solutions ($f = 1_{u \le \lambda}$, σ, M and $f' = 1_{u' \le \lambda}$, σ', M') that correspond to the same initial function $f_0 = 1_{\lambda \ge u_0(x)}$. Since the solution set of the problem K is convex, the half-sum of the two solutions is also a solution corresponding to f_0. Repeating the previous reasoning for the renormalized inequality, we conclude that $\frac{f + f'}{2}$ takes only the values 0 and 1, which implies that $f = f'$; therefore, $u = u'$ a.e. in Q, which proves the uniqueness in Theorem 1.

Note that, according to $f(\mathbf{x}, t, \lambda) = 1_{u(\mathbf{x}, t) \le \lambda}$, the Young measure $\mu_{x, t}$ is a parametrized Dirac measure on $\mathbb{R}_\lambda$ supported by the point $\lambda = u(\mathbf{x}, t)$. In view of the theory of Young measures [9], this means that the sequence u_ε converges to u strongly in $L^1(Q)$ as $\varepsilon \searrow 0$.

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