ENTROPY SOLUTIONS TO A GENUINELY NONLINEAR ULTRAPARABOLIC KOLMOGOROV-TYPE EQUATION

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We consider a non-isotropic convection-diffusion-reaction equation of a very general form, in which the diffusion matrix is nonnegative and may change its rank depending on temporal and spatial variables, and convection and reaction terms may be discontinuous. This equation arises in astrophysics and plasma physics, in fluid dynamics, mathematical biology and financial mathematics. We assume that the equation *a priori* admits the maximum principle and is genuinely nonlinear, and we prove that there exists at least one entropy solution and that the genuinely nonlinear structure of the equation rules out fine oscillatory regimes in entropy solutions. The proofs rely on the method of kinetic equation and on theory of H-measures.

Keywords: Ultra-parabolic equation; Vlasov–Fokker–Planck equation; Entropy solution; Genuine nonlinearity; Non-isotropic diffusion.

1. Introduction

In a space-time layer $\Pi := \mathbb{R}^d_x \times (0,T)$, $T = \text{const} > 0$, we consider the Cauchy problem for the quasilinear equation with partial diffusion and discontinuous convection and reaction terms

$$u_t + \partial_{x_i} a_i(\mathbf{x},t,u) - \partial_{x_i}(a_{ij}(\mathbf{x},t)\partial_{x_j} b(u)) + r(\mathbf{x},t,u) = 0, \tag{1a}$$

endowed with periodic initial data belonging to $L^\infty(\mathbb{R}^d)$ and periodicity conditions

$$u|_{t=0} = u_0(\mathbf{x}), \ \ \mathbf{x} \in \mathbb{R}^d, \tag{1b}$$

$$u(\mathbf{x}+\mathbf{e}_i,t) = u(\mathbf{x},t), \ \ (\mathbf{x},t) \in \Pi. \tag{1c}$$

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In (1) $\mathbf{e}_i$ $(i = 1, ..., d)$ are standard basis vectors in $\mathbb{R}^d$, $u(\mathbf{x}, t)$ is an unknown function; the flux vector $\mathbf{a} := (a_i)$, the diffusion matrix $A := (a_{ij})$, the diffusion function b, and the reaction function r are given and satisfy the conditions

$$a_i, D_{x_i} a_i, r \in L^2_{loc}(\Pi; C^1_{loc}(\mathbb{R}_u)), \quad a_{ij} \in C^2_{loc}(\Pi), \quad b \in C^2_{loc}(\mathbb{R}), \tag{2}$$

$$a_i,\ a_{ij},\ r \text{ are 1-periodic in } \mathbf{x}, \quad A \text{ is symmetric}, \tag{3}$$

$$a_{ij}(\mathbf{x}, t)\xi_i \xi_j \geq 0, \quad b'(u) \geq 0, \quad \forall\, \xi \in \mathbb{R}^d,\ (\mathbf{x}, t) \in \Pi,\ u \in \mathbb{R}. \tag{4}$$

It is assumed that Eq. (1a) a priori admits the maximum principle, i.e., that for its solution (if any) the bound $|u(\mathbf{x}, t)| \leq c_*$ holds a.e. in Π, where the constant c_* depends only on the given data in the problem. The class of conditions providing the maximum principle is very wide, and such a condition may be set explicitly using Ref. 1.

In (1) and further the conventional rule of summation over repeating indexes is in use. Also, further the derivative D_{x_i} is defined by the formula

$$D_{x_i} g(\mathbf{x}, t, u) = (\partial_{x_i} g(\mathbf{x}, t, \lambda))|_{\lambda = u(x,t)}, \quad \forall\, g \in C^1(\Pi \times \mathbb{R}_\lambda).$$

In particular, the derivatives ∂_{x_i} and D_{x_i} are connected via identity

$$\partial_{x_i} g(\mathbf{x}, t, u) = D_{x_i} g(\mathbf{x}, t, u) + \partial_u g(\mathbf{x}, t, u)\partial_{x_i} u.$$

It is supposed that the rank of the diffusion matrix A may be less than the dimension of the space $\mathbb{R}^d_x$ and may vary depending on $\mathbf{x}$ and t. Therefore Eq. (1a) is an ultraparabolic equation. Equations of the form (1a) arise in particle physics, fluid dynamics, combustion theory, mathematical biology, and financial mathematics. They are called *Kolmogorov-type* equations in line with the works of A. N. Kolmogorov relating to problems on stochastic diffusive processes modeling Brownian motion (see in survey Ref. 2). Particular forms of Eq. (1a) have also other names: in problems about nonlinear convection-diffusion-reaction in anisotropic continuous media they are called *Graetz–Nusselt* equations and in studies of the transport of cosmic-rays they are named *Fokker–Planck* equations. They describe, in particular, non-stationary transport of energy or matter in cases, when effects of diffusion in some spatial directions are negligible as compared to convection and reaction. Considerably recently Kolmogorov-type equations have been applied to astrophysical problems: in solar physics with the investigation of acceleration of fast electrons in the solar corona and in space physics with ion acceleration at the solar termination shock and with particle acceleration at astrophysical shocks, including the possibility of second-order Fermi acceleration.[3]

2. Notion of entropy solutions

We are interested in developing the existence and qualitative theory for the Cauchy problem (1) under conditions (2)–(4) and under the assumption that the maximum principle is a priori guaranteed. From the physical point of view, the most appropriate concept of solution to problem (1) is the notion of entropy solutions, since it is consistent with the fundamental fact that in diffusive processes entropy does not decrease.[4,5]

In order to define an entropy solution of problem (1), let us introduce some notation. By Q we denote $\Omega \times (0,T)$, where Ω stands for the unit cube $[0,1)^d$. By $L^p \subset L^p_{loc}(\mathbb{R}^d)$ and $H^{s,p} \subset H^{s,p}_{loc}(\mathbb{R}^d)$ we denote the Banach spaces, which consist of 1-periodic functions and are supplemented with the norms $\|u\|_{L^p} = \|u\|_{L^p(\Omega)}$ and $\|u\|_{H^{s,p}} = \|u\|_{H^{s,p}(\Omega)}$. For $l \geq 0$, let C^l be the closed subspace of $u \in C^l(\mathbb{R}^d)$ such that u is 1-periodic with respect to x_i, $1 \leq i \leq d$. Note that since Eq. (1a) is degenerate, then the gradient $\nabla_x u$ of a possible solution $u \in L^\infty(\Pi)$ may be understood merely in the distributions sense. However, also note that since the matrix A is symmetric and nonnegative, then there is a unique square root $A^{1/2} = \{\alpha_{ij}\}$, which is a symmetric and nonnegative matrix, as well. This and the standard energy estimate[1] yield that a possible solution u of problem (1) should a priori satisfy the bound $\|A^{1/2}\nabla_x \beta(u)\|_{L^2(Q)} \leq c$, where the constant c does not depend on u and $\beta(u) := \int^u \sqrt{b'(s)}ds$.

This means that, although a particular derivative $\partial_{x_i} u$ may not be measurable on Π, the differential expressions of the form $\alpha_{ij}\partial_{x_j}\beta(u)$ involving these derivatives are measurable in Π and integrable with the square in Q, i.e., they belong to $L^2_{loc}(\Pi)$. Therefore the demand of partial integrability of $\nabla_x u$ should be introduced into a notion of entropy solution.

Now we are in a position to define an entropy solution of problem (1).

Definition 2.1. Function $u = u(\mathbf{x}, t)$ is an *entropy solution* of problem (1), if it satisfies the regularity and periodicity conditions $u \in L^\infty(0,T; L^\infty)$ and $\alpha_{ij}\partial_{x_j}\beta(u) \in L^2(0,T;L^2)$, $1 \leq i \leq d$; the entropy inequality

$$\begin{aligned}\partial_t\varphi(u) + \partial_{x_i}q_i(\mathbf{x},t,u) - D_{x_i}q_i(\mathbf{x},t,u) + \varphi'(u)D_{x_i}a_i(\mathbf{x},t,u)\\ -\partial_{x_i}(a_{ij}(\mathbf{x},t)\partial_{x_j}w(u)) + \varphi''(u)\sum_{i=1}^d |\alpha_{ij}(\mathbf{x},t)\partial_{x_j}\beta(u)|^2\\ +\varphi'(u)r(\mathbf{x},t,u) \leq 0 \qquad (5)\end{aligned}$$

in the distributions sense for all functions φ, q_i and w such that $\varphi \in C^2_{loc}(\mathbb{R})$, $\varphi''(u) \geq 0$, $\partial_u q_i(\mathbf{x},t,u) = \varphi'(u)\partial_u a_i(\mathbf{x},t,u)$, and $w'(u) = \varphi'(u)b'(u)$; and the initial data (1b) in the weak sense, i.e., in the sense of the limiting relation $u(\cdot,\tau) \to u_0(\cdot)$ weakly* in L^∞, as $\tau \searrow 0$.

Note that taking $\varphi(u) = \pm u$ in (5) we see that entropy solution satisfies Eq. (1a) in the sense of distributions.

3. Formulation of the main results

Besides conditions (2)–(4), the following *genuine nonlinearity condition* is imposed on the functions a_i, a_{ij}, and b.

Condition 3.1. For a.e. $(\mathbf{x},t) \in \Pi$ the following demand is fulfilled: for all $(\xi,\tau) \in \mathbb{R}^{d+1}$ such that $|\xi|^2 + \tau^2 = 1$ the intersection of the sets $\{\lambda \in \mathbb{R} \mid b'(\lambda)a_{ij}(\mathbf{x},t)\xi_i\xi_j = 0\}$ and $\{\lambda \in \mathbb{R} \mid \tau + \big(\partial_\lambda a_i(\mathbf{x},t,\lambda) + (1/2)b'(\lambda)\partial_{x_j}a_{ij}(\mathbf{x},t)\big)\xi_i = 0\}$ has zero Lebesgue measure.

The following new existence theorem is the first main result of the paper.

Theorem 3.1. *Assume that Eq.* (1a) *is genuinely nonlinear, satisfies conditions* (2)–(4) *and a priori admits the maximum principle. Then problem* (1) *has at least one entropy solution for any given data* $u_0 \in L^\infty$*.*

Also we establish a qualitative property of genuine nonlinearity to rule out fine oscillations developing from initial data, which is the second main result of the paper.

Theorem 3.2. *Assume that Eq.* (1a) *is genuinely nonlinear, satisfies conditions* (2)–(4)*, a priori admits the maximum principle, and is provided with highly oscillatory initial data* $u_0^k \in L^\infty$*,* $k = 1, 2, \dots$ *such that* $u_0^k \to u_0$ *weakly* in* L^∞ *as* $k \nearrow \infty$*.*

Then there exists a subsequence of entropy solutions u^k*, corresponding to initial data* u_0^k*, which tends strongly in* $L^2(0,T;L^2)$ *as* $k \nearrow \infty$ *to an entropy solution* u*, corresponding to initial data* u_0*.*

4. Method of justification of Theorems 3.1 and 3.2

Proofs of Theorems 3.1 and 3.2 rely upon the method of kinetic equation,[6,7] which allows to reduce quasilinear equations and systems to linear scalar equations on "distribution" functions involving additional "kinetic" variables. Alongside this method, the theory of H-measures[8,9] is implemented. In this final section we give a brief explanation of the methodology of the proofs of Theorems 3.1 and 3.2.

First, the kinetic formulation of problem (1) is introduced in the form proposed in Ref. 6. It is linear with respect to the sought function, which is the distribution function $f(\mathbf{x}, t, \lambda) = f_{x,t}(\lambda)$ of the parametrized Dirac measure on $\mathbb{R}_\lambda$ concentrated at the point $\lambda = u(\mathbf{x}, t)$, where $u(\mathbf{x}, t)$ is the entropy solution of problem (1). Notion of entropy solution and the kinetic formulation are equivalent to each other. Linearity of the kinetic formulation becomes possible due to appearance of an additional kinetic variable λ.

Second, the proof of Theorem 3.2 is fulfilled by virtue of the toolbox of the theory of H-measures. The construction of H-measures associated with a weakly convergent subsequence of the distribution functions $f_k(\mathbf{x}, t, \lambda)$, $k = 1, 2, \ldots$ is introduced in the form, which was proposed in Ref. 8 for studying scalar conservation laws. By their nature, H-measures are microlocal defect measures that allow to track evolution of fine oscillatory regimes in the space of time t, positions $\mathbf{x}$ and frequencies ξ. More precisely, for any fixed $\lambda \in \mathbb{R}$ they indicate where in the physical space of time and positions, and at which frequencies in the Fourier space, weakly convergent in L^2_{loc} sequences fail to converge strongly. One of the main properties of H-measures is that they are zero measures if and only if their generating subsequence is strongly convergent. Using the technics of Ref. 10, it is possible to establish a localization principle for the H-measures, i.e., to define a set in the $(\mathbf{x}, t, \xi)$-space such that the H-measures vanish in its complement. This principle combined with the genuine nonlinearity condition immediately imply that the H-measures are equal to zero measure everywhere for almost all $\lambda \in \mathbb{R}$. Hence their generating subsequence is compact, which finishes the proof of Theorem 3.2.

Finally, in order to prove Theorem 3.1, we introduce the well-posed approximation of problem (1), which incorporates a regularizing "small viscosity" coefficient. Availability of such approximation is guaranteed by the well-known theory of parabolic equations.[1] After this, it is sufficient to remark that the approximate problem admits the kinetic formulation of exactly the same form as problem (1). Thus the rest of the proof of Theorem 3.1 is merely the byproduct of the proof of Theorem 3.2.

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